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# Compactness properties of perturbed sub-stochastic semigroups on $L^1(\mu)$ . A preliminary version

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## Abstract

We deal with positive  $c_0$ -semigroups  $\{U(t); t \geq 0\}$  of contractions in  $L^1(\Omega; \mathcal{A}, \mu)$  with generator  $T$  where  $(\Omega; \mathcal{A}, \mu)$  is an abstract measure space and provide a systematic approach of compactness properties of perturbed semigroups  $\{e^{t(T-V)}; t \geq 0\}$  (or their generators) induced by singular and bounded below potentials  $V : (\Omega; \mu) \rightarrow \mathbb{R}$ . The results are precised further for metric measure spaces  $(\Omega, d, \mu)$ . This new theory relies on several ingredients: new a priori estimates peculiar to  $L^1$ -spaces, local weak compactness assumptions on unperturbed operators, “Dunford-Pettis” arguments and the assumption that the sublevel sets  $\Omega_M := \{x; V(x) \leq M\}$  are “thin at infinity with respect to  $\{U(t); t \geq 0\}$ ”. We show also how spectral gaps occur when the sublevel sets are not “thin at infinity”. This formalism suits  $c_0$ -semigroups with integral kernels, for instance most sub-markovian semigroups arising in the theory of Markov processes in general state spaces, and combines intimately the kernel of  $\{U(t); t \geq 0\}$  and the sublevel sets  $\Omega_M$ . Indefinite potentials are also dealt with. We illustrate the relevance of some aspects of this theory by giving new compactness and spectral results on convolution semigroups, magnetic Schrödinger semigroups, weighted Laplacians (in particular the Poincaré inequality for probability measures  $e^{-\Phi(x)}dx$  on  $\mathbb{R}^N$ ) and Witten Laplacians on 1-forms.

## 1 Introduction

We deal with perturbed semigroups in  $L^1(\mu)$  spaces generated by

$$T - V$$

where  $T$  is the generator of a substochastic semigroup and  $V$  is a singular indefinite potential; the meaning of “ $T - V$ ” will be explained below. The object of this paper is to give new functional analytic tools and results on spectral theory (*full discreteness or spectral gaps*) of perturbed semigroups or perturbed generators. In particular, we are concerned with resolvent compactness of  $T - V$  and also with existence of spectral gaps for perturbed generators, i.e.

$$s_{ess}(T - V) < s(T - V)$$

where

$$s(T - V) := \sup \{ \operatorname{Re} \lambda; \lambda \in \sigma(T - V) \}$$

is the spectral bound of  $T - V$  and

$$s_{ess}(T - V) := \sup \{ \operatorname{Re} \lambda; \lambda \in \sigma_{ess}(T - V) \}$$

is the essential spectral bound of  $T - V$ ; we note that  $s(T - V) \in \sigma(T - V)$  by standard theory of positive semigroups. We study also *weak* spectral gaps of perturbed generators, i.e. when the *peripheral* spectrum of  $T - V$ ,

$$\sigma(T - V) \cap \{s(T - V) + i\mathbb{R}\},$$

consists of isolated eigenvalues with finite algebraic multiplicities; this spectral picture does not prevent a priori the existence of sequences of eigenvalues of  $T - V$  with real parts going to  $s(T - V)$  and imaginary parts going to infinity; note that the spectrum  $\sigma(T - V)$  need *not* be real.

Similarly, we study the compactness of perturbed semigroups  $\{e^{t(T-V)}; t \geq 0\}$  and also the existence of spectral gaps, i.e.

$$r_{ess}(e^{t(T-V)}) < r_{\sigma}(e^{t(T-V)})$$

where  $r_{\sigma}(e^{t(T-V)})$  is the spectral radius of  $e^{t(T-V)}$  and

$$r_{ess}(e^{t(T-V)}) := \sup \left\{ |\mu|; \mu \in \sigma_{ess}(e^{t(T-V)}) \right\}$$

is the essential spectral radius of  $e^{t(T-V)}$ , ( $\sigma_{ess}$  refers to essential spectrum). We are mainly interested in the situation where a priori  $T$  is *not* resolvent compact and has *no* spectral gap, i.e. full discreteness or spectral gaps are *induced* by the presence of a potential  $V$ . A new and general theory dealing with many aspects is provided. While most of the known literature on full discreteness or spectral gaps is concerned with hilbertian results and (quite often) by self-adjoint semigroups, we provide here a new point of view relying

on a new circle of ideas peculiar to  $L^1$ -spaces and without any connection with self-adjointness. A systematic approach of the underlying compactness background is given. We show also how this general formalism fits with sub-stochastic semigroups arising in the theory of Markov processes in metric spaces; in particular, various examples from Statistical Mechanics are dealt with. In the case where  $\{e^{tT}; t \geq 0\}$  operates on all  $L^p(\Omega; \mu)$  spaces (e.g. for sub-Markov semigroups) then so does  $\{e^{t(T-V)}; t \geq 0\}$ ; we show then how the  $L^1$  spectral structure of  $\{e^{t(T-V)}; t \geq 0\}$  determines its  $L^p$  spectral structure.

The particular role of positive operators and  $L^1$ -spaces in linear perturbation theory appeared a long time ago in the classical Kato's paper [44] on well-posedness of Kolmogorov's differential equations and also later in the important analytical and probabilistic role played by the so-called Kato class potentials used currently in the theory of Markov processes, see e.g. [1][77][83][21]. Recently, essentially from the beginning of the 2000's, there has been a renewal of interest in perturbation theory of sub-stochastic semigroups in  $L^1$ -spaces where the generator of a sub-stochastic semigroup  $\{e^{tT}; t \geq 0\}$  is perturbed additively by a positive  $T$ -bounded operator  $V$  (see [9] and references therein); this so-called "honesty theory" is motivated by various problems from fragmentation theory or kinetic theory and has a probabilistic counterpart in the concept of non-explosive processes [81]. More recent developments on "honesty theory" are given in [61] while a non-commutative version of [61] (in the Banach space of trace class operators in a Hilbert space), of interest for quantum dynamical semigroups (see e.g. [16] and [25]), is given in [58]; an ultimate extension (with new developments) to general ordered Banach spaces with additive norm on the positive cone is given in [6]. In another direction, it was realized in [59][60][63] that the use of *weak compactness* arguments in  $L^1$  allows a significant generalisation of the (extended) Kato class potentials for Schrödinger-type operators and provide also new (hilbertian) form-bound estimates; these ideas have also useful applications to kinetic theory [62]. The main goal of the present paper is to show how  $L^1$  weak compactness arguments, combined to *new*  $L^1$  *estimates*, allow a systematic spectral analysis of perturbed sub-stochastic semigroups in the case of *negative* unbounded multiplication operators  $-V$ ; we show also how to combine those ideas to another ones (inspired by transport theory [56]) to cover also indefinite potentials  $V = V_+ - V_-$ ; in both situations, local weak compactness tools play a key role. Thus, following the spirit of [59][60][63], this paper continues the exploration of the role of the space  $L^1$  and its weak topology in well-posedness of perturbed evolu-

tion equations, their spectral analysis and also their relevant applications to various sub-markovian equations arising e.g. in classical probability theory.

Let  $(\Omega; \mathcal{A}, \mu)$  be a measure space and let  $\{U(t); t \geq 0\}$  be a positive  $c_0$ -semigroup of contractions on  $L^1(\Omega; \mathcal{A}, \mu)$  with generator  $T$ . We denote by

$$V : (\Omega; \mu) \rightarrow [0, +\infty]$$

(or more generally bounded from below) a measurable potential and denote by  $\{U_V(t); t \geq 0\}$  the (appropriately defined) perturbed semigroup generated by  $T_V := "T - V"$ . The main object of this paper is to give a general and systematic theory of full spectral discreteness or spectral gaps for perturbed generators  $T_V := "T - V"$  or perturbed semigroups  $\{U_V(t); t \geq 0\}$  and to illustrate this theory by significant examples of applied interest. More precisely, we focus on the underlying compactness background. The interplay between the singular potential and the unperturbed semigroup which is in the heart of such compactness or spectral gaps results is finely analyzed in this paper. We give here a point of view on the subject relying on new tools peculiar to  $L^1$  spaces; a completely new formalism is provided. In our general context, the relevant technical tools we need will be different depending on whether we deal with  $T_V$  or  $\{U_V(t); t \geq 0\}$ . Thus, in our study of perturbed generators  $T_V := "T - V"$ , we take advantage of the quite unsuspected fact, in comparison to  $L^2$ -space setting, that  $V$  is always  $T_V$ -bounded in  $L^1$  spaces [66][83]. On the other hand, to study perturbed semigroups  $\{U_V(t); t \geq 0\}$ , we provide two different strategies: the first approach consists in assuming that  $\{U(t); t \geq 0\}$  is norm continuous, in showing the norm continuity of the perturbed semigroup  $\{U_V(t); t \geq 0\}$  and in taking advantage of the properties of perturbed generators and "spectral mapping tools". In the second strategy, we show a "weak type" estimate for almost all  $t > 0$

$$\int_{\{V > M\}} (U_V(t)f)\mu(dx) \leq \frac{c_t \|f\|}{M}, \forall f \in L^1_+(\Omega; \mu), \forall M > 0$$

under the assumption (on the *perturbed* semigroup) that

$$\forall t > 0, \sup_{\varepsilon \in ]0, 1]} \left\| \frac{U_V^*(t + \varepsilon)1 - U_V^*(t)1}{\varepsilon} \right\|_{L^\infty(\Omega; \mu)} < +\infty$$

where  $U_V^*(t)$  is the dual operator of  $U_V(t)$ . (This last assumption is much weaker than a differentiability condition on  $\{U_V(t); t \geq 0\}$  and is satisfied e.g. if

$$\forall f \in L^1(\Omega; \mu), ]0, +\infty[ \ni t \rightarrow \int U_V(t)f \text{ is differentiable}$$

or if

$$]0, +\infty[ \ni t \rightarrow U_V(t) \in \mathcal{L}(L^1(\Omega; \mu)) \text{ is locally lipschitz;}$$

in particular it is satisfied if  $\{U(t); t \geq 0\}$  is holomorphic because  $\{U_V(t); t \geq 0\}$  is then holomorphic too [4][41].) These  $L^1$ -estimates combined to *local* weak compactness assumptions on unperturbed operators, to properties of sub-level sets

$$\Omega_M := \{y; V(y) \leq M\},$$

more precisely their “size at infinity with respect to unperturbed operators”, and to “Dunford-Pettis” arguments, play an important part in our formalism and provide us with new relevant tools in spectral theory of perturbed sub-stochastic semigroups and their generators. Our local  $L^1$  weak compactness assumptions on unperturbed operators are very weak ones and are trivially satisfied by most examples occurring in the literature.

We also deal with *indefinite* potentials  $V = V_+ - V_-$  (with nonnegative  $V_{\pm}$ ), i.e. with operators  $T - (V_+ - V_-)$  considered as perturbed operators

$$T_{V_+} + V_-.$$

This second perturbation theory combines the previous one and different ideas inspired by transport theory [56].

Before explaining more precisely the content of this work, some related information in Hilbert space setting is worth mentioning. According to a classical result going back at least to K. Friedrichs [26], Schrödinger operators  $-\Delta + V$  in  $L^2(\mathbb{R}^N)$  (defined by means of quadratic forms) have fully discrete spectra, or equivalently  $-\Delta + V$  has a compact resolvent, for nonnegative potentials  $V \in L^1_{loc}(\mathbb{R}^N)$  such that  $\lim_{|x| \rightarrow \infty} V(x) = +\infty$ . Of course, it is also known since a long time that this condition is not necessary since F. Rellich [70] already observed for example that for the potential

$$V(x_1, x_2) = x_1^2 x_2^2, \tag{1}$$

$-\Delta + V$  is still resolvent compact in  $L^2(\mathbb{R}^2)$  even if  $V(x_1, x_2)$  fails to go to  $+\infty$  at infinity near the axes. Besides K. Friedrichs [26], the literature on discreteness of the spectrum of Schrödinger operators goes back to A.M. Molchanov [64] and is now considerable; we refer to the survey [74] and also to the more recent paper [53] for more developments. This literature deals with Schrödinger operators on more general non-compact Riemannian manifolds and provides optimal (i.e. necessary and sufficient) conditions of discreteness in terms of Wiener capacity of suitable sets. Such sharp results are not always of simple practical use but sufficient or necessary conditions

in terms of measures are also available; among the various statements we note A.M. Molchanov's necessary condition of discreteness

$$\int_{B(x,r)} V(y) dy \rightarrow +\infty \quad \text{as } x \rightarrow \infty \quad (2)$$

(which is also sufficient in one dimension) and also the sufficient criterion:

**Theorem 1** ([74] Corollary 10.2, p. 268). *We assume that for any  $M > 0$  the sublevel set  $\Omega_M := \{y; V(y) \leq M\}$  is “thin at infinity” in the sense that for some  $r > 0$*

$$|B(x, r) \cap \Omega_M| \rightarrow 0 \quad \text{as } x \rightarrow \infty \quad (3)$$

*where  $B(x, r)$  is the ball centered at  $x$  with radius  $r$  (and  $||$  refers to Lebesgue measure). Then  $-\Delta + V$  in  $L^2(\mathbb{R}^N)$  has a discrete spectrum.*

In ([28] Lemma 5 and Remark 2) it is observed that the sublevel sets of a nonnegative function  $V$  are “thin at infinity” if and only if

$$\int_{B(x,r)} \frac{1}{1+V(y)} dy \rightarrow 0 \quad \text{as } x \rightarrow \infty; \quad (4)$$

the argument relies on the simple double inequality (for arbitrary  $M > 0$ )

$$\begin{aligned} \frac{1}{1+M} |B(x, r) \cap \Omega_M| &\leq \int_{B(x,r)} \frac{1}{1+V(y)} dy \\ \int_{B(x,r)} \frac{1}{1+V(y)} dy &\leq |B(x, r) \cap \Omega_M| + \frac{1}{1+M} |B(0, r)|. \end{aligned}$$

One realizes then that Theorem 1 was already known in 1978 under Assumption (4) [11]; it seems that this has not been noticed in the literature on the subject. One sees also how the necessary condition (2) follows from “thinness at infinity” of sublevel sets  $\Omega_M$  since

$$\begin{aligned} |B(0, r)| &= |B(x, r)| = \int_{B(x,r)} \frac{\sqrt{1+V(y)}}{\sqrt{1+V(y)}} dy \\ &\leq \left( \int_{B(x,r)} \frac{1}{1+V(y)} dy \right)^{\frac{1}{2}} \left( \int_{B(x,r)} (1+V(y)) dy \right)^{\frac{1}{2}} \end{aligned}$$

and then

$$\int_{B(x,r)} V(y) dy \geq -|B(0, r)| + \frac{|B(0, r)|^2}{\int_{B(x,r)} \frac{1}{1+V(y)} dy}.$$

More recently, it was shown in [48] that  $T - V$  is resolvent compact in  $L^2(\mathbb{R}^N)$  when  $T$  is the relativistic  $\alpha$ -stable operator

$$T = -(-\Delta + m^\frac{2}{\alpha})^\frac{\alpha}{2} + m$$

provided that  $\lim_{|x| \rightarrow \infty} V(x) = +\infty$ . This result was extended in [84] (for sublevels sets  $\Omega_M$  having finite measure only) to much more general symmetric Markov generators in  $L^2(\Omega; \mu)$  satisfying the so-called *intrinsic super Poincaré inequality* and such that the Markov semigroup has a density with respect to  $\mu$ . The proof given by the authors is however quite involved and combines various technical arguments; shortly after, a simpler proof was given in [78] and other developments, still for self-adjoint operators in Hilbert spaces, were also given in [28][51]. Even the finiteness assumption on the measure of the sublevels sets  $\Omega_M$  has been dropped. For instance, we find in [78] that if  $T$  is a self-adjoint operator in  $L^2(\Omega; \mu)$  such that  $\{e^{tT}; t \geq 0\}$  is an ultracontractive semigroup in the sense that (for  $t > 0$ )  $e^{tT}$  maps  $L^2(\Omega; \mu)$  into  $L^\infty(\Omega; \mu)$  then  $T - V$  is resolvent compact in  $L^2(\Omega; \mu)$  provided that  $V \in L^1_{loc}(\mathbb{R}^N)$  and  $V \geq 0$  is such that its sublevels sets are  $r$ -polynomially thin (for some  $r > 0$ ), i.e. for any  $R > 0$

$$\int_{\Omega_M} |\Omega_M \cap B(x; R)|^r \mu(dx) < +\infty.$$

We note that in  $\mathbb{R}^N$ ,  $r$ -polynomially thin set is necessarily thin at infinity in the sense (3) (see [28] Lemma 7). On the other hand, it is known (see e.g. [8]) that the discreteness of the spectrum of the *magnetic* Schrödinger operator in  $L^2(\mathbb{R}^N)$  is strongly connected to that of the Schrödinger operator via the diamagnetic inequality. Finally, there exists also an important literature on *Poincaré* (or spectral gap) *inequalities* for Markov semigroups arising in Probability and Statistical Mechanics

$$var_\mu(f) := \int_\Omega f^2 d\mu - \left( \int_\Omega f d\mu \right)^2 \leq c(A^\frac{1}{2} f, A^\frac{1}{2} f), \quad f \in D(A^\frac{1}{2}),$$

of interest e.g. for exponential trend to equilibrium, where  $(\Omega, \mu)$  is a probability space,  $A$  is a nonnegative self-adjoint operator in  $L^2(\Omega, \mu)$ ,  $1 \in D(A)$  and  $A1 = 0$ ; (such inequalities are sometimes derived from Log Sobolev (or Gross) inequalities; see [33][72][3][37][85]). Note that a spectral gap expresses simply that 0, the bottom of  $\sigma(A)$ , is an isolated eigenvalue with finite algebraic multiplicity; as such, the notion of a spectral gap is meaningful in much more general situations but, of course, cannot be formulated



in terms of variance inequality. Actually, this notion amounts to strict positivity of the bottom of the essential spectrum  $\sigma_{ess}(A)$ ; we refer to [68][55] for the location of essential spectra of Schrödinger operators  $-\Delta + V$  in  $L^2(\mathbb{R}^N)$  when the sublevel sets of  $V$  are not “thin at infinity”. We point out that all the results above are hilbertian; in particular no  $L^1$  compactness result nor spectral gap result in  $L^1$  space can a priori be derived from the literature above. We mention also the paper [29] on spectral gaps for bounded positive operators in  $L^p$ -spaces with  $p > 1$  and various applications. This very brief overview shows various contexts where full discreteness or spectral gaps are worth studying.

As far as potentials bounded from below are concerned, we provide here *a new point of view on discreteness and on spectral gaps* relying on a different circle of ideas. Neither selfadjointness nor  $L^2$  spaces play a role in our approach. By contrast, the functional space  $L^1(\Omega; \mu)$ , the positivity of the unperturbed semigroup under consideration (this could be relaxed by relying on its modulus [45], see Section 6), the fact that  $V$  is always  $T_V$ -bounded in  $L^1$  spaces and the “weak type” estimate

$$\int_{\{V > M\}} (U_V(t)f)\mu(dx) \leq \frac{c_t \|f\|}{M}, \forall f \in L^1_+(\Omega; \mu), \forall M > 0,$$

valid under suitable assumptions (or the norm continuity of  $\{U_V(t); t \geq 0\}$  valid under other suitable assumptions) provide us with the starting point of a completely new formalism. By complementing these  $L^1$ -estimates by *local* weak compactness assumptions on  $(\lambda - T)^{-1}$  or  $\{U(t); t \geq 0\}$  and taking advantage of “Dunford-Pettis” arguments, we can build a general theory where various related functional analytic results are given; the stability of essential spectra by weakly compact perturbations (see e.g. [49]), combined to the above ingredients, turns out to be the right tool to deal with spectral gaps. We provide thus a pure  $L^1$  theory on full discreteness or spectral gaps where most of the results are new. Moreover, our construction has the advantage of being conceptually simple and self-contained. We note that in the special case where the semigroup operates in all  $L^p$  spaces ( $1 \leq p < +\infty$ ), e.g. for sub-Markov semigroups, our  $L^1$  compactness results imply, by interpolation, compactness results in  $L^p$  for  $p > 1$ , providing us e.g. with hilbertian results, while converse statements are not true a priori, see e.g. Markov semigroups generated by weighted Laplacians which are compact in  $L^p$  for  $p > 1$  but fail to be so in  $L^1$  [18] Section 4.3. We point out that our primary goal here is not a priori to obtain hilbertian results by means of  $L^1$  techniques; it is rather to build and explore an  $L^1$  theory for its own sake and this program is undertaken here for the first time. We

note also that our  $L^p$  results ( $p > 1$ ) are new since they are *deduced* from an  $L^1$  theory; in a sense,  $L^p$  spectral theory of perturbed submarkovian semigroups becomes a sub-product of the  $L^1$  theory which acquires thus a *special status*. Our results are given for general measure spaces  $(\Omega; \mathcal{A}, \mu)$  and precised further for metric measure spaces  $(\Omega, d, \mu)$ . In addition, a special section is also devoted to more specific results on convolution semigroups on euclidean spaces (e.g. on subordinate Brownian semigroups) because of their importance in applications. Finally, various related examples from Statistical Mechanics are revisited: thus, Markov semigroups stemming from Dirichlet forms in weighted  $L^2$  spaces turn out to be unitarily equivalent to usual Schrödinger semigroups (with potentials) that can be dealt with by our  $L^1$  formalism while various spectral results on Witten Laplacians on 1-forms (of interest for Helffer- Sjöstrand's covariance formula) are also given. Our approach of the subject suits semigroups exhibiting integral kernels; this happens under our local *weak-compactness* assumptions (see Remark 15 (ii)); e.g. ultracontractive symmetric Markov semigroups for separable measure spaces meet our conditions. Thus, for  $L^1$  spaces over metric measure spaces  $(\Omega, d, \mu)$ , our compactness or spectral gap results combine intimately integral kernels of unperturbed semigroups and sublevel sets  $\Omega_M := \{y; V(y) \leq M\}$  of the singular potential. For instance, this provides us with sufficient conditions in terms of heat kernel and sublevel sets of  $\frac{1}{4} |\nabla \Phi|^2 - \frac{1}{2} \Delta \Phi$  for a probability measure  $e^{-\Phi(x)} dx$  on  $\mathbb{R}^N$  to satisfy the Poincaré inequality. A last section is devoted to *indefinite* potentials

$$V = V_+ - V_-.$$

Actually, all the paper shows how fruitful is the  $L^1$  treatment of perturbed sub-Markov semigroups provided some reasonable upper estimate on their integral kernels is available.

Of course, various kinds of upper estimates of transition kernels appear in the literature on Markov processes in metric spaces. For instance, the Heat kernel associated to the Laplace Beltrami operator on non-compact complete Riemannian manifolds  $(\Omega, d, \mu)$  of dimension  $n$  ( $d$  is the geodesic distance and  $\mu$  is the Riemannian volume) with Ricci curvature bounded below and having the so-called “bounded geometry” (see [18] p. 172) satisfies a Gaussian estimate for each  $t > 0$

$$p_t(x, y) \leq C_t^1 \exp\left(-\frac{d(x, y)^2}{C_t^2}\right), \quad (5)$$

see e.g. [18][30][31]. However, Brownian motions on some fractal spaces lead

to transition kernels with sub-Gaussian estimates

$$p_t(x, y) \leq \frac{C}{t^{\frac{\alpha}{\beta}}} \exp\left(-\left(\frac{d^\beta(x, y)}{Ct}\right)^{\frac{1}{\beta-1}}\right) \quad (6)$$

where  $\alpha > 0$  is the Hausdorff dimension and  $\beta > 2$  is “a walk dimension”, see e.g. [10]. On the other hand, the study of kernel estimates for non local Dirichlet forms, in connection with Markov processes with jumps, developed also in the last decades and typical kernel estimates of jump Markov semigroups are polynomial

$$p_t(x, y) \leq \frac{C}{t^{\frac{\alpha}{\beta}}} \left(1 + \frac{d(x, y)}{t^{\frac{1}{\beta}}}\right)^{-(\alpha+\beta)}, \quad (7)$$

see e.g. [39]. (We refer to [7][32] for much more information on the very rich subject of “Heat kernels”.) This “ubiquity” of integral kernels suggests that there is a room for a general theory of compactness (and spectral) properties for a large class of perturbed  $c_0$ -semigroups which is the object of this work.

We outline now some of our main results:

In Section 2, we consider a measure space  $(\Omega; \mathcal{A}, \mu)$  and a positive  $c_0$ -semigroup of contractions  $\{U(t); t \geq 0\}$  on  $L^1(\Omega; \mathcal{A}, \mu)$  with generator  $T$ . Let

$$V : (\Omega; \mu) \rightarrow [0, +\infty]$$

be measurable and let  $\{U_V(t); t \geq 0\}$  be the contraction semigroup defined by

$$U_V(t)f := \lim_{n \rightarrow +\infty} e^{t(T-V_n)}f$$

where  $V_n := V \wedge n$ . We note that this semigroup need not a priori be strongly continuous at the origin but we restrict ourselves to the case where it is so and denote by  $T_V$  its generator. We show that  $T_V$  has a compact resolvent if and only if for all  $M > 0$  the operator

$$(\lambda - T_V)^{-1} : L^1(\Omega; \mu) \rightarrow L^1(\Omega_M; \mu) \text{ is weakly compact}$$

(we mean the operator:  $f \in L^1(\Omega; \mu) \rightarrow [(\lambda - T_V)^{-1}f]_{|\Omega_M} \in L^1(\Omega_M; \mu)$ ) where  $\Omega_M := \{y; V(y) \leq M\}$  are the sublevel sets of  $V$ . It follows from the domination  $(\lambda - T_V)^{-1} \leq (\lambda - T)^{-1}$  that a sufficient condition for  $T_V$  to be resolvent compact is that

$$(\lambda - T)^{-1} : L^1(\Omega; \mu) \rightarrow L^1(\Omega_M; \mu) \text{ is weakly compact;} \quad (8)$$

under a technical additional assumption (which is satisfied e.g. in denumerable state spaces), we show that (8) is also necessary.

If

$$\{U_V(t); t \geq 0\} \text{ is norm continuous}$$

i.e.

$$]0, +\infty[ \ni t \rightarrow U_V(t) \in \mathcal{L}(L^1(\Omega; \mu))$$

is continuous in operator norm, then (8) implies the stronger result that the perturbed semigroup  $\{U_V(t); t \geq 0\}$  is compact on  $L^1(\Omega; \mu)$ . The question about when this norm continuity assumption is satisfied is also dealt with: we show first the stability estimate

$$\sup_{t \leq C} \|e^{t(T-V_n)}f - U_V(t)f\| \leq e^C \|[V - V_n](1 - T_V)^{-1}f\|, \quad \forall f \in L^1_+(\Omega; \mu)$$

for arbitrary  $C > 0$  (where  $\{[V - V_n](1 - T_V)^{-1}\}_n$  is a sequence of bounded operators going *strongly* to zero as  $n \rightarrow +\infty$ ) which has its own interest and which implies that  $\{U_V(t); t \geq 0\}$  is norm continuous provided that  $\{U(t); t \geq 0\}$  is norm continuous and

$$\|[V - V_n](1 - T_V)^{-1}\|_{\mathcal{L}(L^1(\Omega; \mu))} \rightarrow 0 \text{ as } n \rightarrow +\infty;$$

in particular, if  $(1 - T_V)^{-1}$  is an integral operator with kernel  $G_V(x, y)$  then  $\{U_V(t); t \geq 0\}$  is norm continuous provided that  $\{U(t); t \geq 0\}$  is norm continuous and

$$\sup_{y \in \Omega} \int_{\{V \geq n\}} G_V(x, y) V(x) \mu(dx) \rightarrow 0 \text{ as } n \rightarrow +\infty. \quad (9)$$

On the other hand, if for all  $M > 0$

$$U(t) : L^1(\Omega; \mu) \rightarrow L^1(\Omega_M; \mu) \text{ is weakly compact } (t > 0) \quad (10)$$

and if the dual operator  $U_V^*(t)$  satisfies

$$\forall t > 0, \sup_{\varepsilon \in ]0, 1]} \left\| \frac{U_V^*(t + \varepsilon)1 - U_V^*(t)1}{\varepsilon} \right\|_{L^\infty(\Omega; \mu)} < +\infty \quad (11)$$

(the latter holds e.g. if

$$\forall f \in L^1(\Omega; \mu), ]0, +\infty[ \ni t \rightarrow \int U_V(t)f \text{ is differentiable}$$

or if

$$]0, +\infty[ \ni t \rightarrow U_V(t) \in \mathcal{L}(L^1(\Omega; \mu)) \text{ is locally lipschitz})$$

then  $\{U_V(t); t \geq 0\}$  is a compact semigroup on  $L^1(\Omega; \mu)$ . Thus the norm continuity condition on  $\{U_V(t); t \geq 0\}$  (e.g. (9)) and (11) (which are not comparable, see Remark 4(ii) below) provide us with two independent approaches of the compactness of the perturbed semigroup  $\{U_V(t); t \geq 0\}$ . Fortunately, all the assumptions above are satisfied if  $\{U(t); t \geq 0\}$  is *holomorphic* because  $\{U_V(t); t \geq 0\}$  is then holomorphic too [4][41]. Condition (11) is of particular interest since we show that it implies the “weak-type” estimate for almost all  $t > 0$

$$\int_{\{V > M\}} (U_V(t)f)\mu(dx) \leq \frac{c_t \|f\|}{M}, \forall f \in L^1_+(\Omega; \mu), \forall M > 0$$

which turns out to be a key tool in the study of spectral gaps for  $\{U_V(t); t \geq 0\}$ . We note that (11) is much weaker than a differentiability condition on the perturbed semigroup  $\{U_V(t); t \geq 0\}$  and does not even imply its norm continuity, see Remark 4(ii).

We show also that  $T_V$  is resolvent compact if (10) is satisfied. Moreover, Assumption (10) is shown to be stable by subordination; the proof of this relies on the fact that a strong integral (*not* necessarily a Bochner integral) on a finite measure space of a strongly measurable bounded operator-valued mapping with values in  $W(E, F)$  (the space of weakly compact operators between Banach spaces  $E$  and  $F$ ) belongs to  $W(E, F)$  [75]; see also [57] when  $F$  is an  $L^1(\nu)$ -space. This result has significant applications e.g. to subordinate Brownian semigroups on euclidean spaces (see below).

In the special case where  $\{U(t); t \geq 0\}$  operates on all  $L^p(\Omega; \mu)$  spaces then so does  $\{U_V(t); t \geq 0\}$  (we note them respectively  $\{U_p(t); t \geq 0\}$  and  $\{U_{pV}(t); t \geq 0\}$  when acting in  $L^p(\Omega; \mu)$ ) and then various compactness results in  $L^p(\Omega; \mu)$  are also obtained by interpolation; in particular, under Assumption (8) *only*, if  $\{U_2(t); t \geq 0\}$  is symmetric in  $L^2(\Omega; \mu)$  then  $\{U_{pV}(t); t \geq 0\}$  is a compact semigroup in  $L^p(\Omega; \mu)$  for  $p > 1$  (but is not a priori so in  $L^1(\Omega; \mu)$ ).

Because of their applied interest, we devote Section 3 to specific results on convolution semigroups (related to Lévy processes) on euclidean spaces. We show first that if  $h \in L^1(\mathbb{R}^N)$  and if

$$H : \varphi \in L^1(\mathbb{R}^N) \rightarrow \int_{\mathbb{R}^N} h(x-y)\varphi(y)dy$$

then, for a Borel set  $\Xi \subset \mathbb{R}^N$ ,  $H : L^1(\mathbb{R}^N) \rightarrow L^1(\Xi)$  is compact if and only if

$$\sup_{y \in \mathbb{R}^N} \int_{\Xi \cap \{|x| \geq c\}} h(x-y)dx \rightarrow 0 \text{ as } c \rightarrow \infty$$

and the latter condition is satisfied if  $\Xi$  is “thin at infinity” in the sense (3). This allows us to deal with convolution semigroups

$$U(t) : f \in L^1(\mathbb{R}^N) \rightarrow \int f(x-y)m_t(dy) \in L^1(\mathbb{R}^N)$$

where  $\{m_t\}_{t \geq 0}$  are Borel sub-probability measures on  $\mathbb{R}^N$  such that  $m_0 = \delta_0$  (the Dirac measure at zero),  $m_t * m_s = m_{t+s}$  and  $m_t \rightarrow m_0$  vaguely as  $t \rightarrow 0_+$ . The sub-probability measures  $\{m_t\}_{t \geq 0}$  are characterized by

$$\widehat{m}_t(\zeta) := (2\pi)^{-\frac{N}{2}} \int e^{-i\zeta \cdot x} m_t(dx) = (2\pi)^{-\frac{N}{2}} e^{-tF(\zeta)}, \quad \zeta \in \mathbb{R}^N$$

where  $F(\zeta)$  is the so-called characteristic exponent; (see e.g. [42] Chapter 3). The resolvent of the generator  $T$  is also a convolution with a measure  $m^\lambda$

$$(\lambda - T)^{-1}f = \int f(x-y)m^\lambda(dy)$$

where  $m^\lambda = \int_0^{+\infty} e^{-\lambda t} m_t dt$  ( $\lambda > 0$ ) is a vaguely convergent integral such that

$$\widehat{m^\lambda}(\zeta) = \int_0^{+\infty} e^{-\lambda t} \widehat{m}_t(\zeta) dt = \frac{1}{\lambda + F(\zeta)}.$$

Thus, if  $m^\lambda$  is a function (i.e. is absolutely continuous with respect to Lebesgue measure), in particular if  $e^{-tF(\zeta)} \in L^1(\mathbb{R}^N)$  for  $t > 0$ , then  $T_V$  has a compact resolvent provided that the sublevel sets  $\Omega_M$  are “thin at infinity” in the sense (3). We show also the compactness of perturbed semigroups in  $L^p(\mathbb{R}^N)$  ( $p \geq 1$ ) for *all* subordinate Brownian semigroups; this covers for example the relativistic  $\alpha$ -stable semigroup generated by

$$T = -(-\Delta + m^{\frac{2}{\alpha}})^{\frac{\alpha}{2}} + m \quad (0 < \alpha < 2, \quad m \geq 0).$$

In Section 4, we deal with sub-stochastic semigroups in  $L^1$  spaces over metric measure spaces, i.e. metric spaces  $(\Omega, d)$  endowed with a Borel measure  $\mu$  which is finite on bounded Borel subsets of  $\Omega$ . This framework is motivated by Markov processes in metric spaces, (see e.g. [32] and references therein). The existence of a metric  $d$  allows to precise further some of the results in Section 2. We show that if (11) is satisfied (e.g. if  $\{U(t); t \geq 0\}$  is holomorphic) and if  $U(t)$  is such that  $U(t) : L^1(\Omega; \mu) \rightarrow L^1(\Xi; \mu)$  is weakly compact for any bounded Borel set  $\Xi \subset \Omega$  then  $\{U_V(t); t \geq 0\}$  is a compact semigroup in  $L^1(\Omega; \mu)$  provided that for some  $x_0 \in \Omega$

$$\lim_{C \rightarrow +\infty} \sup_{y \in \Omega} \int_{\{x \in \Omega_M; d(x, x_0) \geq C\}} p_t(x, y) \mu(dx) = 0$$

where  $p_t(x, y)$  is the kernel of  $U(t)$  (the existence of this kernel is a consequence of the weak compactness assumption, see [24] p. 508); we express this by saying that the sublevel sets  $\Omega_M$  are “thin at infinity with respect to  $\{U(t); t \geq 0\}$ ”. In particular, if

$$v(r) := \sup_{x \in \Omega} \mu(B(x, r)) < \infty \quad \forall r \geq 0$$

and if  $p_t(., .)$  satisfies an estimate of the form

$$p_t(x, y) \leq f_t(d(x, y))$$

where  $f_t : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is nonincreasing and such that (for large  $r$ ) the function  $r \rightarrow f_t(r)v(r+1)$  is nonincreasing and integrable at infinity then the sublevel sets  $\Omega_M$  are “thin at infinity with respect to  $\{U(t); t \geq 0\}$ ” if they are “thin at infinity” in the sense there exists a point  $\bar{y} \in \Omega$  such that for any  $R > 0$

$$\mu\{\Omega_M \cap B(y, R)\} \rightarrow 0 \quad \text{as } d(y, \bar{y}) \rightarrow +\infty;$$

thus if we consider e.g. the typical examples of kernel estimates (5), (6) and (7) occurring in the study Markov processes in metric spaces, one easily sees which volume growth  $r \rightarrow v(r)$  (for large  $r$ ) is compatible with the above assumption on  $r \rightarrow f_t(r)v(r+1)$ . Other statements in terms of the kernel of  $(1 - T)^{-1}$  are also given when  $\{U_V(t); t \geq 0\}$  is norm continuous.

In Section 5, we relax the assumption that the sublevel sets  $\Omega_M$  are “thin at infinity with respect to  $\{U(t); t \geq 0\}$ ”. We show, under suitable kernel estimates involving the sublevel sets  $\Omega_M$  that the essential spectral radius  $r_{ess}(U_V(t))$  of the semigroup  $U_V(t)$  is less than its spectral radius, i.e.  $U_V(t)$  exhibits a *spectral gap*. To this end, we take full advantage of the stability of essential spectra by weakly compact perturbations in  $L^1$  spaces (see e.g. [49]). More precisely, if (11) is satisfied (e.g. if  $\{U(t); t \geq 0\}$  is holomorphic) and if for some  $t > 0$ ,  $U(t) : L^1(\Omega) \rightarrow L^1(\Xi)$  is weakly compact for any bounded Borel set  $\Xi$  and the kernel  $p_t(x, y)$  of  $U(t)$  satisfies the estimate

$$\sup_{M > 0} \lim_{C \rightarrow +\infty} \sup_{y \in \Omega} \int_{\{x \in \Omega_M; d(x, x_0) \geq C\}} p_t(x, y) \mu(dx) < e^{\lambda_V t}$$

(for some  $x_0 \in \Omega$ ) where  $\lambda_V := s(T_V)$  is the spectral bound of  $T_V$  then  $\omega_{ess} < \lambda_V$  where  $\omega_{ess}$  is the *essential type* of  $\{U_V(t); t \geq 0\}$ ; note that

$$e^{\omega_{ess} t} = r_{ess}(U_V(t)) \quad \forall t > 0,$$

see e.g. [65] p. 74. We observe that  $\lambda_V$  is also the type of  $\{U_V(t); t \geq 0\}$  since the latter is a positive semigroup in  $L^1$  space ([65] Theorem 1.1 p. 334) and then

$$e^{\lambda_V t} = r_\sigma(U_V(t)) \leq \|U_V(t)\|_{\mathcal{L}(L^1(\Omega))} \leq \|U(t)\|_{\mathcal{L}(L^1(\Omega))} = \sup_{y \in \Omega} \int_{\Omega} p_t(x, y) \mu(dx).$$

One can avoid the use of the (a priori unknown) parameter  $\lambda_V$  and obtain a slightly different result formulated as an *alternative*: Indeed, using  $\lambda_1$  (the spectral bound of  $T$ ) instead of  $\lambda_V$  we show that if

$$\sup_{M > 0} \lim_{C \rightarrow +\infty} \sup_{y \in \Omega} \int_{\{x \in \Omega_M; d(x, x_0) \geq C\}} p_t(x, y) \mu(dx) < e^{\lambda_1 t}$$

then either  $\lambda_V < \lambda_1$  or  $\lambda_V = \lambda_1$  and  $\omega_{ess} < \lambda_V$  where  $\omega_{ess}$  is the essential type of  $\{U_V(t); t \geq 0\}$ . In particular, if  $\{U(t); t \geq 0\}$  is a stochastic semigroup i.e. is mass preserving on the positive cone and if

$$\sup_{M > 0} \lim_{C \rightarrow +\infty} \sup_{y \in \Omega} \int_{\{x \in \Omega_M; d(x, x_0) \geq C\}} p_t(x, y) \mu(dx) < 1$$

then either  $\lambda_V < 0$  or  $\lambda_V = 0$  and  $\omega_{ess} < 0$ . In the case where  $\{U(t); t \geq 0\}$  operates on all  $L^p(\Omega)$  ( $p \geq 1$ ) and if we denote by  $\lambda_p$  (resp.  $\lambda_{pV}$ ) the spectral bound of the generator of  $\{U_p(t); t \geq 0\}$  (resp. of  $\{U_{pV}(t); t \geq 0\}$ ) then, under the estimate

$$\sup_{M > 0} \lim_{C \rightarrow +\infty} \sup_{y \in \Omega} \int_{\{x \in \Omega_M; d(x, x_0) \geq C\}} p_t(x, y) \mu(dx) < e^{p\lambda_p t},$$

we have either  $\lambda_{pV} < \lambda_p$  or  $\lambda_{pV} = \lambda_p$  and  $\omega_{pess} < \lambda_{pV}$  where  $\omega_{pess}$  is the essential type of  $\{U_{pV}(t); t \geq 0\}$ .

We can also avoid such alternatives for symmetric semigroups. We show then that  $\{U_V(t); t \geq 0\}$  has a spectral gap (i.e.  $\omega_{ess} < \lambda_V$ ) provided that

$$\sup_{M > 0} \lim_{C \rightarrow +\infty} \sup_{y \in \Omega} \int_{\{x \in \Omega_M; d(x, x_0) \geq C\}} p_t(x, y) \mu(dx) < e^{\hat{\lambda} t}$$

where  $\hat{\lambda} := \lim_{r \rightarrow \infty} \hat{\lambda}_r$  (increasing limit)

$$-\hat{\lambda}_r := \inf_{\varphi \in D_r, \|\varphi\|_{L^2(\Omega; \mu)} = 1} \left\| \sqrt{-T_2} \varphi \right\|_{L^2(\Omega; \mu)}^2 + \int_{B(x_0, r)} V(x) |\varphi(x)|^2 \mu(dx),$$

$$D := \left\{ \varphi \in D(\sqrt{-T_2}), \int V(x) |\varphi(x)|^2 \mu(dx) < +\infty \right\}$$



and  $D_r$  is the subspace of  $D$  consisting of those elements with supports included in  $B(x_0, r)$ . Note that  $\widehat{\lambda}_r \leq \widehat{\lambda}$  and that the left hand side of the above strict inequality depends on the values of the potential “at infinity” while the right hand side depends on its values at finite distance only. We point out that there exists an important literature on spectral gaps in  $L^2$  setting in terms of *Poincaré inequalities*, (such inequalities are also related to Log Sobolev (or Gross) inequalities, see e.g. [33][72][3][37][85][14][29]). We provide here a new point of view in  $L^1$  spaces in terms of kernels estimates of unperturbed semigroups and sublevel sets of the potential.

We also study *weak* spectral gaps for generators  $T_V$ . Indeed, we show that if the kernel  $G_1(x, y)$  of  $(1 - T)^{-1}$  satisfies the estimate

$$\sup_{M>0} \lim_{C \rightarrow +\infty} \sup_{y \in \Omega} \int_{\{x \in \Omega_M; d(x, x_0) \geq C\}} G_1(x, y) \mu(dx) < \frac{1}{1 - \lambda_V}$$

(for some  $x_0 \in \Omega$ ) then the *peripheral* spectrum of  $T_V$ , i.e.

$$\sigma(T_V) \cap \{\lambda_V + i\mathbb{R}\},$$

consists of isolated eigenvalues with finite algebraic multiplicities (see Theorem 33 for a more precise statement). This spectral picture does not prevent a priori the existence of sequences of eigenvalues of  $T_V$  with real parts going to  $s(T_V)$  and imaginary parts going to infinity. However, if  $\{U_V(t); t \geq 0\}$  is norm continuous then we show that above estimate implies the much stronger conclusion that this semigroup has a spectral gap.

Sections 6, 7 and 8 illustrate the practical usefulness of some of previous functional analytic results by providing new results in three directions of applied interest: Magnetic Schrödinger operators (Section 6), weighted Laplacians (Section 7) and Witten Laplacians on 1-forms (Section 8); we revisit some important examples of the subject by exploiting in particular  $L^1$  techniques.

Section 6 is devoted to magnetic Schrödinger semigroups; (besides its mathematical and physical interest, this class of semigroups illustrates significantly the fact that, in the above general theory, the positivity assumption on the unperturbed semigroup could be relaxed by using domination arguments; in principle, this strategy could even be used in full generality by exploiting the existence of a *modulus* of the unperturbed semigroup i.e. a minimal dominating positive contraction semigroup [45]). According to a classical result, see e.g. [8], the spectrum of the magnetic Schrödinger operator in  $L^2$  is discrete if this is the case for the Schrödinger operator without magnetic potential; the pointwise diamagnetic inequality being the key

point. Thus, the  $L^2$  compactness results for Schrödinger operators are automatically translated into  $L^2$  compactness results for magnetic Schrödinger operators. By following our  $L^1$  approach and using the diamagnetic inequality, we give a compactness result in  $L^1$  setting when the sublevel sets  $\Omega_M$  are “thin at infinity”. This  $L^1$  point of view complements the known hilbertian results on discreteness of magnetic Schrödinger operators. On the other hand, as far as we know, the situation where the sublevel sets  $\Omega_M$  are not “thin at infinity” has not been dealt with yet; we show here the existence of a spectral gap for magnetic Schrödinger operators under a condition involving the heat kernel and the sublevel sets of the potential, (see Theorem 45).

In Section 7, we deal with some aspects of weighted Laplacians, (see e.g. [18][31][38] for the interest of the subject); in particular, we revisit some problems which were considered in [38] in connection with Fokker-Planck operators. We consider the weighted Laplacian

$$\Delta^\mu := \frac{1}{h^2} \operatorname{div}(h^2 \nabla) = \Delta + 2 \frac{\nabla h \cdot \nabla}{h}$$

which is (minus) the self-adjoint operator in  $L^2(\mathbb{R}^N; \mu(dx))$  associated to the Dirichlet form  $\int_{\mathbb{R}^N} |\nabla \varphi|^2 \mu(dx)$  where  $\mu(dx) = h^2(x)dx$  with  $h \in C^2(\mathbb{R}^N)$  and  $h(x) > 0 \forall x \in \mathbb{R}^N$ . (We have restricted ourselves to  $\Delta$  for simplicity but more general elliptic operators with smooth coefficients can be dealt with similarly.) This operator is unitarily equivalent to the Schrödinger operator  $\Delta - \frac{\Delta h}{h}$  on  $L^2(\mathbb{R}^N; dx)$ . In particular, if  $h(x) = e^{-\frac{\Phi}{2}(x)}$  where  $\Phi$  is a real  $C^2$  function on  $\mathbb{R}^N$  then  $\frac{\Delta h}{h} = \frac{1}{4} |\nabla \Phi(x)|^2 - \frac{1}{2} \Delta \Phi(x)$ . Then, assuming that  $\frac{1}{4} |\nabla \Phi(x)|^2 - \frac{1}{2} \Delta \Phi(x)$  is bounded from below, we can exploit our previous  $L^1$  results on Schrödinger operators to give new (compactness) results on the subject. Thus, we consider the (non-convex) potential

$$\Phi(x) = \frac{1}{h} \sum_{j=1}^N \left( \frac{\lambda}{12} x_j^4 + \frac{\nu}{2} x_j^2 \right) + \frac{1}{h} \frac{I}{2} \sum_{j=1}^N |x_j - x_{j+1}|^2$$

with the convention  $x_{N+1} = x_1$  where  $h > 0$ ,  $\lambda > 0$ ,  $\nu < 0$ ,  $I > 0$  (which appears e.g. in [36][43]) and show that  $\Delta - (\frac{1}{4} |\nabla \Phi(x)|^2 - \frac{1}{2} \Delta \Phi(x))$  generates a (holomorphic) compact semigroup in  $L^1(\mathbb{R}^N; dx)$ .

When  $\Phi$  is a uniformly strictly convex potential then by a classical result of D. Bakry and M. Emery (see e.g. [72] Théorème 3.1.29, p. 50) a logarithmic Sobolev inequality holds implying in particular the spectral gap (or Poincaré) inequality; we complement this result by showing that actually

$\Delta - (\frac{1}{4}|\nabla\Phi(x)|^2 - \frac{1}{2}\Delta\Phi(x))$  generates a (holomorphic) compact semigroup in  $L^1(\mathbb{R}^N; dx)$ .

We consider also the case of (nonpositive) polynomial potential

$$\Phi(x) = - \sum_{|\alpha| \leq C} c_\alpha x_1^{2\alpha_1} x_2^{2\alpha_2} \dots x_N^{2\alpha_N}, \quad (c_\alpha > 0)$$

where  $\bar{\alpha}_i > 0 \forall i$  for at least one multi-index  $\bar{\alpha}$ ; it is known (see [38] Theorem 11.10 (ii), p. 120) that  $\Delta - (\frac{1}{4}|\nabla\Phi(x)|^2 - \frac{1}{2}\Delta\Phi(x))$  is resolvent compact in  $L^2(\mathbb{R}^N; dx)$ ; we show here that actually  $\Delta - (\frac{1}{4}|\nabla\Phi(x)|^2 - \frac{1}{2}\Delta\Phi(x))$  generates a (holomorphic) compact semigroup in  $L^1(\mathbb{R}^N; dx)$ .

The homogeneous (nonnegative) case

$$\Phi(x) = \sum_{|\alpha|=r} c_\alpha x_1^{2\alpha_1} x_2^{2\alpha_2} \dots x_N^{2\alpha_N}, \quad (c_\alpha > 0)$$

is also dealt with but only in the simplest “elliptic” case  $\nabla\Phi(x) \neq 0 \forall x \neq 0$ , (we refer to [38] for a systematic analysis of polynomial potentials in  $L^2$  setting).

We deal also with spectral gaps when  $\frac{1}{4}|\nabla\Phi(x)|^2 - \frac{1}{2}\Delta\Phi(x)$  is bounded from below; we give a sufficient condition for the existence of a spectral gap for  $\Delta^\mu$  in  $L^2(\mathbb{R}^N; \mu(dx))$  under kernel estimates involving the sublevel sets of  $\frac{1}{4}|\nabla\Phi(x)|^2 - \frac{1}{2}\Delta\Phi(x)$ . In particular, if  $\frac{1}{4}|\nabla\Phi(x)|^2 - \frac{1}{2}\Delta\Phi(x) \geq 0$  and  $e^{-\Phi(x)} \in L^1(\mathbb{R}^N; dx)$ , the existence of a spectral gap for  $\Delta^\mu$  is guaranteed under the condition

$$\sup_{M>0} \lim_{C \rightarrow +\infty} \sup_{y \in \mathbb{R}^N} \int_{\{x \in \Omega_M; |x| \geq C\}} \frac{1}{(4\pi t)^{\frac{N}{2}}} \exp(-\frac{|x-y|^2}{4t}) dx < 1.$$

Thus, this condition provides us with a sufficient criterion in terms of sublevel sets of  $\frac{1}{4}|\nabla\Phi(x)|^2 - \frac{1}{2}\Delta\Phi(x)$  for a probability measure  $e^{-\Phi(x)}dx$  on  $\mathbb{R}^N$  to satisfy the *Poincaré inequality*.

In Section 8, we deal with Witten Laplacians, i.e. weighted Hodge Laplacians, on 1-forms associated to the Witten Complex (i.e. the exterior differential  $d$  of the De Rham Complex is replaced by  $d_\Phi := e^{-\frac{\Phi}{2}(x)} d e^{\frac{\Phi}{2}(x)}$  where  $\Phi$  is a suitable smooth function; see e.g. [79][43] and [37] Chapter 2). The Witten Laplacian on 0-forms is unitarily equivalent to

$$\Delta_\Phi^{(0)} = \Delta^{(0)} + \frac{1}{4}|\nabla\Phi|^2 - \frac{1}{2}\Delta\Phi$$

(where  $\Delta^{(0)} = -\Delta$ ) while the Witten Laplacian on 1-forms is unitarily equivalent to

$$\Delta_\Phi^{(1)} = \Delta_\Phi^{(0)} \otimes Id + Hess\Phi$$

where 1-forms are identified with vector functions; both Laplacians are non-negative and the spectral bottom of  $\Delta_{\Phi}^{(0)}$  is zero when  $e^{-\Phi(x)}dx$  is a probability measure. The interest of Witten Laplacians in Statistical Mechanics stems in particular from the beautiful Helffer- Sjöstrand's covariance formula

$$\int (f(x) - \langle f \rangle)(g(x) - \langle g \rangle) e^{-\Phi(x)} dx = \int \left( (\Delta_{\Phi}^{(1)})^{-1} df, dg \right) e^{-\Phi(x)} dx,$$

where  $\langle f \rangle = \int f(x) e^{-\Phi(x)} dx$  (see [79][43] and [37] Chapter 2). The invertibility of  $\Delta_{\Phi}^{(1)}$  is of course a key point; actually, it suffices that the restriction of  $\Delta_{\Phi}^{(1)}$  to exact 1-forms be invertible; (see [43] for the details). By combining  $L^1$  results and hilbertian tools (Glazman's Lemma) we show here that if  $\Phi$  is convex (no strict convexity is needed) then the *essential* lower bound of  $\Delta_{\Phi}^{(0)}$  is less than or equal to that of  $\Delta_{\Phi}^{(1)}$ ; in particular  $\Delta_{\Phi}^{(1)}$  is resolvent compact if  $\Delta_{\Phi}^{(0)}$  is. We show also, for convex  $\Phi$ , that if  $\Delta_{\Phi}^{(0)}$  has spectral gap and if  $Hess\Phi$  is not degenerate, i.e. its lowest eigenvalue is not identically zero, then the spectral bottom of  $\Delta_{\Phi}^{(1)}$  is strictly bigger than that of  $\Delta_{\Phi}^{(0)}$  (and consequently  $\Delta_{\Phi}^{(1)}$  is invertible if  $e^{-\Phi(x)}dx$  is a probability measure). Regardless of any convexity assumption, we show also that if  $\lambda_{\Phi}$  is the lowest eigenvalue of  $Hess\Phi$ , if  $\frac{1}{4}|\nabla\Phi|^2 - \frac{1}{2}\Delta\Phi + \lambda_{\Phi}$  is lower bounded and if  $\Delta_{\Phi}^{(0)} + \lambda_{\Phi}$  is resolvent compact then  $\Delta_{\Phi}^{(1)}$  is also resolvent compact; in particular  $\Delta_{\Phi}^{(1)}$  is resolvent compact if the sublevel sets of  $\frac{1}{4}|\nabla\Phi|^2 - \frac{1}{2}\Delta\Phi + \lambda_{\Phi}$  are "thin at infinity". When such sublevel sets are not "thin at infinity", we show the existence of a spectral gap for  $\Delta_{\Phi}^{(1)}$  in terms of the heat kernel and these sublevel sets, (see Theorem 64).

In the last section (Section 9) we come back to the general theory and deal with operators " $T - (V_+ - V_-)$ " with *indefinite* potentials  $V = V_+ - V_-$  and consider them as perturbed operators

$$T_{V_+} + V_-$$

provided that  $V_-$  is  $T_{V_+}$ -bounded. The tools behind their treatment are different from the ones above used for  $T_{V_+}$ . According to [22] (see e.g. [9] Chapter 5),

$$T_{V_+} + V_- : D(T_{V_+}) \rightarrow L^1(\Omega; \mu)$$

generates a positive semigroup  $\{W(t); t \geq 0\}$  if and only if

$$\lim_{\lambda \rightarrow +\infty} r_{\sigma} [V_-(\lambda - T_{V_+})^{-1}] < 1.$$

We show that if  $\{e^{tT}; t \geq 0\}$  is holomorphic then so is  $\{W(t); t \geq 0\}$ . We show also that if  $\{e^{tT}; t \geq 0\}$  is norm continuous then so is  $\{W(t); t \geq 0\}$  provided that

$$\sup_{y \in \Omega} \int_{\{V_{\pm} \geq j\}} G_{V_{\pm}}(x, y) V_{\pm}(x) \mu(dx) \rightarrow 0 \text{ as } j \rightarrow +\infty$$

where  $G_{V_{\pm}}(x, y)$  is the kernel of  $(1 - T_{V_{\pm}})^{-1}$ . In both cases,  $\{W(t); t \geq 0\}$  is shown to be a compact semigroup under (8) (where  $\Omega_M$  are the sublevel sets of  $V_{\pm}$ ). We study also *stability* of essential types (or essential spectral bounds). Indeed, we assume that  $\Omega$  is a locally compact metric space endowed with a locally finite Borel measure  $\mu \geq 0$  and that  $\Omega = \cup_n \Xi_n$  where  $\Xi_n$  is a nondecreasing sequence of compact subsets. We show that  $\{e^{t(T_{V_{\pm}} + V_{\mp}}); t \geq 0\}$  and  $\{e^{tT_{V_{\pm}}}; t \geq 0\}$  have the same essential spectrum and consequently the same essential type provided that

$$\lim_{n \rightarrow \infty} \sup_{y \in \Omega} \int_{\Xi_n^c} V_{\mp}(x) G_{V_{\pm}}(x, y) \mu(dx) = 0;$$

this stability of essential type implies that  $\{e^{t(T_{V_{\pm}} + V_{\mp}}); t \geq 0\}$  has a spectral gap if  $\{e^{tT_{V_{\pm}}}; t \geq 0\}$  has since the type of the latter is less than or equal to that of the former. We deal also with  $L^p$  spaces when  $\{e^{tT}; t \geq 0\}$  operates on all  $L^p$  spaces and restrict ourselves for the sake of simplicity (see Remark 82) to the case where  $\{e^{tT}; t \geq 0\}$  is *symmetric*, i.e. when  $\{e^{tT}; t \geq 0\}$  coincides with its dual  $\{e^{tT'}; t \geq 0\}$  (on  $L^{\infty}(\Omega)$ ) on the space  $L^1(\Omega) \cap L^{\infty}(\Omega)$ ; then both  $\{e^{tT_{V_{\pm}}}; t \geq 0\}$  and  $\{W(t); t \geq 0\}$  interpolate on all  $L^p(\Omega)$  ( $1 \leq p < \infty$ ) providing positive strongly continuous semigroups  $\{e^{tT_{pV_{\pm}}}; t \geq 0\}$  and  $\{e^{tA_p}; t \geq 0\}$  in  $L^p(\Omega)$  where  $T_{2V_{\pm}}$  and  $A_2$  are self-adjoint in  $L^2(\Omega)$ . We note that  $V_{\mp}$  is *not* a priori  $T_{pV_{\pm}}$ -bounded in  $L^p(\Omega)$ ; it is shown in [63] that in  $L^2(\Omega)$ ,  $V_{\mp}$  is *form-bounded* with respect to  $-T_{2V_{\pm}}$  with relative form-bound less than or equal to  $\lim_{\lambda \rightarrow +\infty} r_{\sigma} [V_{\mp}(\lambda - T_{V_{\pm}})^{-1}]$  and (minus)  $A_2$  turns out to be a *form-sum*

$$-A_2 = (-T_{2V_{\pm}}) \dot{+} (-V_{\mp}).$$

Finally, we show by interpolation arguments how  $\{e^{tA_p}; t \geq 0\}$  inherits from  $\{e^{t(T_{V_{\pm}} + V_{\mp}}); t \geq 0\}$  various compactness results and spectral stability results.

An abridged version of this paper will be published soon.

## 2 Compactness properties of sub-stochastic semigroups in $L^1(\Omega; \mu)$

In all this section  $(\Omega; \mathcal{A}, \mu)$  denotes a measure space and  $\{U(t); t \geq 0\}$  is a positive  $c_0$ -semigroup of contractions on  $L^1(\Omega; \mu)$  (i.e. a sub-stochastic  $c_0$ -semigroup) with generator  $T$ . Let

$$V : (\Omega; \mu) \rightarrow [0, +\infty]$$

be a measurable function. (Indefinite potentials will be dealt with in Section 9.) Let  $V_n := V \wedge n$  and  $\{e^{t(T-V_n)}; t \geq 0\}$  be the  $c_0$ -semigroup generated by  $T - V_n$ . It is elementary to see that  $e^{t(T-V_{n+1})}f \leq e^{t(T-V_n)}f \ \forall f \in L^1_+(\Omega; \mu)$  so that

$$U_V(t)f := \lim_{n \rightarrow +\infty} e^{t(T-V_n)}f$$

defines a semigroup. The semigroup  $\{U_V(t); t \geq 0\}$  is a priori strongly continuous for  $t > 0$  only. We say that  $V$  is admissible for  $\{U(t); t \geq 0\}$  if  $\{U_V(t); t \geq 0\}$  is a  $c_0$ -semigroup, i.e. is strongly continuous at the origin. In such a case,  $T_V$ , the generator of  $\{U_V(t); t \geq 0\}$ , is an extension of  $T - V : D(T) \cap D(V) \rightarrow L^1(\Omega; \mu)$ . A sufficient condition of admissibility is that  $D(T) \cap D(V)$  be dense in  $L^1(\Omega; \mu)$  ([83] Proposition 2.9). Actually the above considerations hold also in all  $L^p$  spaces. On the other hand, the following known result is peculiar to  $L^1$ -setting [66][83]; for reader's convenience, we recall briefly its proof (as given in [83] Lemma 4.1) in a slightly different form.

**Lemma 2** *Let  $V \geq 0$  be admissible for  $\{U(t); t \geq 0\}$ . Then  $D(T_V) \subset D(V)$  and  $V$  is  $T_V$ -bounded.*

*Proof:* For a bounded potential  $W$  and  $f \in D(T) \cap L^1_+(\Omega; \mu)$  we have

$$\begin{aligned} \frac{d}{dt} \|e^{-\lambda t} U_W(t)f\| &= \frac{d}{dt} \int e^{-\lambda t} U_W(t)f \, d\mu = \int \frac{d}{dt} [e^{-\lambda t} U_W(t)f] \, d\mu \\ &= \int (T - \lambda - W) [e^{-\lambda t} U_W(t)f] \, d\mu \\ &= \int (T - \lambda) [e^{-\lambda t} U_W(t)f] \, d\mu - \int W [e^{-\lambda t} U_W(t)f] \, d\mu \\ &\leq -e^{-\lambda t} \|W U_W(t)f\| \end{aligned}$$

and consequently

$$\int_0^{+\infty} e^{-\lambda t} \|W U_W(t)f\| \, dt \leq - \int_0^{+\infty} \frac{d}{dt} \|e^{-\lambda t} U_W(t)f\| \, dt = \|f\|.$$

Thus

$$\int_0^{+\infty} e^{-\lambda t} \|V_n U_{V_m}(t)f\| dt \leq \|f\|, \quad \forall m \geq n$$

since  $U_{V_m}(t) \leq U_{V_n}(t)$ . Letting  $m \rightarrow +\infty$ , by monotone (decreasing) convergence we get  $\int_0^{+\infty} e^{-\lambda t} \|V_n U_V(t)f\| dt \leq \|f\|$  and then, by monotone (increasing) convergence, we obtain  $\int_0^{+\infty} e^{-\lambda t} \|V U_V(t)f\| dt \leq \|f\|$  which is nothing but  $\|V(\lambda - T_V)^{-1}f\| \leq \|f\|$  for  $f \in D(T) \cap L_+^1(\Omega; \mu)$ . Finally the density of  $D(T) \cap L_+^1(\Omega; \mu)$  in  $L_+^1(\Omega; \mu)$  and the fact that  $L^1(\Omega; \mu) = L_+^1(\Omega; \mu) - L_+^1(\Omega; \mu)$  show that  $V(\lambda - T_V)^{-1}$  is a bounded operator or equivalently  $V$  is  $T_V$ -bounded. ■

It seems that we cannot hope a priori (see Remark 5 below) that  $U_V(t)$  maps continuously  $L^1(\Omega; \mu)$  into  $D(V)$  for  $t > 0$ . We show however a crucial “weak type” estimate under a suitable assumption on the perturbed semigroup  $\{U_V(t); t \geq 0\}$  itself.

**Lemma 3** *We assume that  $L^1(\Omega; \mu)$  is separable and that  $\{U_V(t); t \geq 0\}$  is such that*

$$\forall t > 0, \sup_{\varepsilon \in ]0,1]} \left\| \frac{U_V^*(t+\varepsilon)1 - U_V^*(t)1}{\varepsilon} \right\|_{L^\infty(\Omega; \mu)} < +\infty \quad (12)$$

where  $U_V^*(t)$  is the dual operator of  $U_V(t)$ . Then, for almost all  $t > 0$ , there exists a positive constant  $c_t$  such that

$$\int_{\{V > M\}} (U_V(t)f)\mu(dx) \leq \frac{c_t \|f\|}{M}, \quad \forall f \in L_+^1(\Omega; \mu), \quad \forall M > 0. \quad (13)$$

In particular, (12) is satisfied if

$$\forall f \in L^1(\Omega; \mu), \quad ]0, +\infty[ \ni t \rightarrow \int U_V(t)f \text{ is differentiable} \quad (14)$$

or if

$$]0, +\infty[ \ni t \rightarrow U_V(t) \in \mathcal{L}(L^1(\Omega; \mu)) \text{ is locally lipschitz.}$$

*Proof:* As in the proof of Lemma 2, we have

$$\frac{d}{dt} \|U_{V_n}(t)f\| \leq -\|V_n U_{V_n}(t)f\|$$

so that, for any  $0 \leq a < b < +\infty$ ,

$$\int_a^b \|V_n U_{V_n}(s)f\| ds \leq \|U_{V_n}(a)f\| - \|U_{V_n}(b)f\|$$

and, for  $n \geq m$ ,

$$\int_a^b \|V_m U_{V_n}(s)f\| ds \leq \|U_{V_n}(a)f\| - \|U_{V_n}(b)f\|.$$

Letting  $n \rightarrow +\infty$  gives

$$\int_a^b \|V_m U_V(s)f\| ds \leq \|U_V(a)f\| - \|U_V(b)f\|$$

so that letting  $m \rightarrow +\infty$  we get by monotone convergence theorem

$$\int_a^b \|V U_V(s)f\| ds \leq \|U_V(a)f\| - \|U_V(b)f\|$$

showing in particular that  $U_V(s)f \in D(V)$  for almost all  $s$ . We note that  $s \rightarrow \|V U_V(s)f\|$  is lower semicontinuous (and thus measurable) as an increasing limit of continuous functions and is locally Lebesgue integrable. Hence, for any  $f \in L^1(\Omega; \mu)$ ,

$$[0, +\infty[ \ni t \rightarrow \int_0^t \|V U_V(s)f\| ds \text{ is differentiable a.e.}$$

(on the set  $E_f$  of Lebesgue points of  $s \rightarrow \|V U_V(s)f\|$ ) with derivative  $\|V U_V(t)f\|$ , see e.g. [73] Théorème 6.1.23, p. 30. Taking  $f \in L_+^1(\Omega; \mu)$ ,  $t \in E_f$

$$\begin{aligned} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \|V U_V(s)f\| ds &\leq \frac{1}{\varepsilon} [\|U_V(t)f\| - \|U_V(t+\varepsilon)f\|] \\ &= \int \frac{U_V^*(t)1 - U_V^*(t+\varepsilon)1}{\varepsilon} f \leq c_t \|f\| \end{aligned}$$

where  $c_t := \sup_{\varepsilon \in ]0,1]} \varepsilon^{-1} \|U_V^*(t+\varepsilon)1 - U_V^*(t)1\|_{L^\infty(\Omega; \mu)}$ , so that letting  $\varepsilon \rightarrow 0$  we get

$$\|V U_V(t)f\| \leq c_t \|f\| \quad (t \in E_f).$$

If  $L^1(\Omega; \mu)$  is separable then so is  $L_+^1(\Omega; \mu)$ . Let  $D \subset L_+^1(\Omega; \mu)$  be a denumerable set dense in  $L_+^1(\Omega; \mu)$  and let  $E := \cap_{f \in D} E_f$ . Finally

$$\|V U_V(t)f\| \leq c_t \|f\|, \quad f \in D, \quad t \in E$$

where the complement set (in  $\mathbb{R}_+$ )  $E^c = \cup_{f \in D} E_f^c$  has zero Lebesgue measure. It follows that for all  $f \in D$



$$M \int_{\{V>M\}} U_V(t)f \leq \int_{\{V>M\}} VU_V(t)f \leq \|VU_V(t)f\| \leq c_t \|f\|$$

and then

$$\int_{\{V>M\}} (U_V(t)f)\mu(dx) \leq \frac{c_t \|f\|}{M}, \quad f \in L_+^1(\Omega; \mu)$$

since  $D$  is dense in  $L_+^1(\Omega; \mu)$  and  $U_V(t)$  is a bounded operator on  $L^1(\Omega; \mu)$ .

The differentiability of  $]0, +\infty[ \ni t \rightarrow \int U_V(t)f$  for all  $f \in L^1(\Omega; \mu)$  amounts to

$$\forall t > 0, \lim_{\varepsilon \rightarrow 0} \frac{U_V^*(t+\varepsilon)1 - U_V^*(t)1}{\varepsilon} \text{ exists}$$

in the weak star topology of  $L^\infty(\Omega; \mu)$  which in turn implies the boundedness of  $\varepsilon^{-1} \|U_V^*(t+\varepsilon)1 - U_V^*(t)1\|_{L^\infty(\Omega; \mu)}$  for  $\varepsilon \in ]0, 1]$  by the uniform boundedness principle. Finally

$$\begin{aligned} \|U_V^*(t+\varepsilon)1 - U_V^*(t)1\|_{L^\infty(\Omega; \mu)} &\leq \|U_V^*(t+\varepsilon) - U_V^*(t)\|_{\mathcal{L}(L^\infty(\Omega; \mu))} \\ &= \|U_V(t+\varepsilon) - U_V(t)\|_{\mathcal{L}(L^1(\Omega; \mu))} \end{aligned}$$

show the last claim. ■

**Remark 4** (i) The separability of  $L^1(\Omega; \mu)$  holds e.g. if  $\mu$  is a  $\sigma$ -finite regular Borel measure on a separable metric space  $\Omega$ .

(ii) We note that the condition that  $]0, +\infty[ \ni t \rightarrow U_V(t) \in \mathcal{L}(L^1(\Omega; \mu))$  be locally lipschitz is weaker than a differentiability condition on the perturbed semigroup  $\{U_V(t); t \geq 0\}$  because the differentiability of a bounded semigroup  $\{S(t); t \geq 0\}$  in a Banach space  $X$  is equivalent to global Lipschitz conditions

$$\forall \varepsilon > 0, \exists C_\varepsilon > 0; \|S(t) - S(s)\|_{\mathcal{L}(X)} \leq C_\varepsilon |t - s|, \quad \forall t, s \geq \varepsilon,$$

see e.g. [40] Lemma 2.1. Note that (14) is also much weaker than a differentiability condition of the perturbed semigroup  $\{U_V(t); t \geq 0\}$ . For instance, if we consider the translation semigroup  $U(t)f = f(x-t)$  in  $L^1(\mathbb{R}, dx)$  then (14) is satisfied by the (non differentiable) perturbed semigroup

$$U_V(t)f = e^{-\int_{x-t}^x V(s)ds} f(x-t)$$

provided that  $V$  is differentiable and

$$\forall t > 0, y \rightarrow V'(y+t)e^{-\int_y^{y+t} V(s)ds} - V^2(y+t)e^{-\int_y^{y+t} V(s)ds} \text{ is bounded.}$$

Actually, in this example,  $\{U_V(t); t \geq 0\}$  is not even norm continuous; this shows that a priori there is no connection between Conditions (12) or (14) and norm continuity of the perturbed semigroup. Such conditions deserve to be investigated more deeply for their own sake. In particular, sufficient conditions in terms of  $\{U(t); t \geq 0\}$  and  $V$  would be very important; note that  $\{U_V(t); t \geq 0\}$  is holomorphic if  $\{U(t); t \geq 0\}$  is [4][41]. A natural condition to be investigated is of course the differentiability of the perturbed semigroup  $\{U_V(t); t \geq 0\}$ ; we mention that in general the differentiability property of a semigroup is not stable by bounded perturbations [71], see however [23][40] for the positive results in this direction; the case of unbounded perturbations seems to be open.

(iii) We note finally that if  $U_V(t)$  is an integral operator with kernel  $p_t^V(x, y)$  then  $U_V^*(t)1 = \int_{\Omega} p_t^V(x, \cdot) \mu(dx)$  and (12) is satisfied if

$$]0, +\infty[ \ni t \rightarrow \int_{\Omega} p_t^V(x, \cdot) \mu(dx) \in L^{\infty}(\Omega; \mu) \text{ is locally lipschitz.}$$

**Remark 5** Despite the denseness of  $D$  in  $L_+^1(\Omega; \mu)$ , it is unclear whether the estimate  $\|VU_V(t)f\| \leq c_t \|f\|$ ,  $f \in D$ ,  $t \in E$  implies that  $VU_V(t)$  is a bounded operator on  $L^1(\Omega; \mu)$  for  $t \in E$  which is a stronger conclusion than (13).

We give now:

**Lemma 6** Let  $\{V(t); t \geq 0\}$  be a  $c_0$ -semigroup on  $L^1(\Omega; \mu)$  with generator  $G$ . If the resolvent  $(\lambda - G)^{-1}$  is a weakly compact operator for some (or equivalently all)  $\lambda \in \rho(G)$  then it is a compact operator for all  $\lambda \in \rho(G)$ .

*Proof:* The resolvent identity

$$(\lambda - G)^{-1} - (\mu - G)^{-1} = (\mu - \lambda)(\lambda - G)^{-1}(\mu - G)^{-1}; \quad \lambda, \mu \in \rho(G)$$

shows that the weak compactness of  $(\lambda - G)^{-1}$  implies the weak compactness of  $(\mu - G)^{-1}$ . By the classical Dunford-Pettis' theorem (see e.g. [2] Corollary 5.88, p. 344) the product of two weakly compact operators on  $L^1(\Omega; \mu)$  is a compact operator so that

$$\|(\lambda - G)^{-1} - (\mu - \lambda)(\lambda - G)^{-1}(\mu - G)^{-1}\| = \|(\mu - G)^{-1}\| \rightarrow 0 \text{ as } \mu \rightarrow +\infty$$

shows that  $(\lambda - G)^{-1}$  is a compact operator. ■

We note that if  $(\lambda - T)^{-1}$  is weakly compact (or equivalently compact) then  $(\lambda - T_V)^{-1}$  is also weakly compact by domination. Similarly if  $U(t)$  is

compact for  $t > 0$  then so is  $U_V(t)$ . Thus, in all this section, it is understood that  $(\lambda - T)^{-1}$  and  $U(t)$  are not (weakly) compact in  $L^1(\Omega; \mu)$ . We are now ready to show:

**Theorem 7** *Let  $\{U(t); t \geq 0\}$  be a sub-stochastic  $c_0$ -semigroup on  $L^1(\Omega; \mu)$  with generator  $T$  and let  $V : (\Omega; \mu) \rightarrow [0, +\infty]$  be admissible for  $\{U(t); t \geq 0\}$ . Then  $T_V$  is resolvent compact if and only if for all  $M > 0$*

$$(\lambda - T_V)^{-1} : L^1(\Omega; \mu) \rightarrow L^1(\Omega_M; \mu) \text{ is weakly compact.} \quad (15)$$

*A sufficient condition for (15) to hold is that*

$$(\lambda - T)^{-1} : L^1(\Omega; \mu) \rightarrow L^1(\Omega_M; \mu) \text{ is weakly compact.} \quad (16)$$

*Proof:* According to Lemma 6, it suffices to show that  $T_V$  is resolvent weakly compact. Let  $s(T_V)$  be the spectral bound of  $T_V$ . Let  $f = (\lambda - T_V)^{-1}g$  with  $\lambda > s(T_V)$  ( $g \in B$ ) where  $B$  is the unit ball of  $L^1(\Omega; \mu)$ . Since  $D(T_V) \subset D(V)$  and  $V$  is  $T_V$ -bounded (Lemma 2) then there exists a constant  $c > 0$  such that  $\|Vf\| \leq c \|g\|$  so that

$$\begin{aligned} M \int_{\{V(x) \geq M\}} |f(x)| \mu(dx) &\leq \int_{\{V(x) \geq M\}} V(x) |f(x)| \mu(dx) \\ &\leq \int V(x) |f(x)| \mu(dx) \leq c, \quad \forall g \in B \end{aligned}$$

so that  $\int_{\{V(x) \geq M\}} |f(x)| \mu(dx) \rightarrow 0$  as  $M \rightarrow +\infty$  uniformly in  $g \in B$ . Thus we have decomposed  $f = (\lambda - T_V)^{-1}g$  as  $f1_{\Omega_M} + f1_{\Omega_M^c}$  where  $f1_{\Omega_M^c}$  can be made as small in  $L^1$ -norm as we want (uniformly in  $g \in B$ ) and  $f1_{\Omega_M}$  is a relatively weakly compact set by (15). This shows the first claim. Finally, the domination  $(\lambda - T_V)^{-1} \leq (\lambda - T)^{-1}$  shows that (16) implies (15). ■

Under an additional technical assumption (which is satisfied e.g. in denumerable state spaces, see Remark 9), we can show that (15) and (16) are equivalent. Indeed, a bounded measurable function  $\varphi : (\Omega; \mu) \rightarrow \mathbb{R}$  is said to operate on  $D(T)$  if  $\varphi\psi \in D(T)$  for any  $\psi \in D(T)$  and  $\psi \in D(T) \rightarrow \varphi\psi \in D(T)$  is continuous for the graph norm.

**Theorem 8** *We assume that for each  $M$  there exist  $\widetilde{M} > M$  and a bounded measurable function  $\varphi_M : (\Omega; \mu) \rightarrow \mathbb{R}$  operating on  $D(T)$  with support in  $\Omega_{\widetilde{M}}$  and equal to one on  $\Omega_M$ . If  $T_V$  is resolvent compact then for all  $M$ ,  $(\lambda - T)^{-1} : L^1(\Omega; \mu) \rightarrow L^1(\Omega_M; \mu)$  is compact.*

*Proof:* Let  $(\lambda - T_V)^{-1} : L^1(\Omega; \mu) \rightarrow L^1(\Omega; \mu)$  be compact. Consider the equation

$$\lambda f - Tf = h; \quad \|h\|_{L^1(\Omega; \mu)} \leq 1.$$

Note first that  $\left\{f; \|h\|_{L^1(\Omega; \mu)} \leq 1\right\}$  is bounded in  $D(T)$  endowed with the graph norm. Let  $\varphi_M : (\Omega; \mu) \rightarrow \mathbb{R}$  operating on  $D(T)$  with support in  $\Omega_{\widetilde{M}}$  and equal to one on  $\Omega_M$ . Then

$$\lambda f \varphi_M - \varphi_M T f + \varphi_M V f = \varphi_M h + \varphi_M V f$$

and  $k := f \varphi_M$  satisfies the equation

$$\lambda k - Tk + Vk = \varphi_M h + \varphi_M V f + \varphi_M T f - Tk.$$

We note that by assumption  $\varphi_M h + \varphi_M V f + \varphi_M T f - Tk$  lives in a bounded subset of  $L^1(\Omega; \mu)$ . The set  $\left\{f \varphi_M; \|h\|_{L^1(\Omega; \mu)} \leq 1\right\}$  is relatively compact in  $L^1(\Omega; \mu)$  by the compactness of  $(\lambda - T_V)^{-1}$  and finally  $\left\{f; \|h\|_{L^1(\Omega; \mu)} \leq 1\right\}$  is relatively compact in  $L^1(\Omega_M; \mu)$  since  $\varphi_M$  is equal to one on  $\Omega_M$ , i.e.  $(\lambda - T)^{-1} : L^1(\Omega; \mu) \rightarrow L^1(\Omega_M; \mu)$  is compact. ■

**Remark 9** Let  $L^1(\mathbb{N}; \mu)$  endowed with the counting measure  $\mu$ . Consider an infinite matrix  $\{a_{i,j}, i, j \in \mathbb{N}\}$  such that  $a_{i,i} \leq 0$ ,  $a_{i,j} \geq 0$  for  $i \neq j$  and  $\sum_i a_{i,j} = 0$ . Let  $T^0$  be the multiplication operator by  $\{a_{i,i}\}_{i \in \mathbb{N}}$  with domain

$$\left\{\{u_i\}_i \in l^1(\mathbb{N}); \{a_{i,i}u_i\}_i \in l^1(\mathbb{N})\right\}$$

and  $\widehat{A} : D(T^0) \rightarrow l^1(\mathbb{N})$  with  $(\widehat{A}u)_i = \sum_{j \neq i} a_{i,j}u_j$ . Note that  $\sum_{i \neq j} a_{i,j} = -a_{j,j}$ . If  $\lim_{\lambda \rightarrow +\infty} \sup_j \sum_{i \neq j} \frac{a_{i,j}}{\lambda - a_{j,j}} < 1$  then  $T := T^0 + \widehat{A} : D(T^0) \rightarrow l^1(\mathbb{N})$  generates a stochastic (i.e. mass preserving on the positive cone)  $c_0$ -semigroup  $\{U(t); t \geq 0\}$  on  $l^1(\mathbb{N})$  (see e.g. [44]). Then  $D(T) = D(T^0)$  is obviously invariant under the multiplication by any sequence  $\{z_i\}_i \in l^\infty(\mathbb{N})$ . Consider now a nonnegative  $V = \{V_i\}_i$ . For any  $\widetilde{M} > M$ , the bounded sequence  $\varphi = \{\varphi_i\}_{i \in \mathbb{N}}$  defined by  $\varphi_i = 1$  if  $V_i \leq \widetilde{M}$  and  $\varphi_i = 0$  otherwise, satisfies the conditions on  $\varphi_M$  in Theorem 8.

We show now how to reach the conclusions of Theorem 7 (and also more stronger ones) by different means.

**Theorem 10** *Let  $\{U(t); t \geq 0\}$  be a sub-stochastic  $c_0$ -semigroup on  $L^1(\Omega; \mu)$  with generator  $T$ . Let  $V : (\Omega; \mu) \rightarrow [0, +\infty]$  be admissible for  $\{U(t); t \geq 0\}$ . We assume that for  $M > 0$  and  $t > 0$*

$$U(t) : L^1(\Omega; \mu) \rightarrow L^1(\Omega_M; \mu) \text{ is weakly compact.} \quad (17)$$

*Then:*

- (i)  $T_V$  is resolvent compact.
- (ii) If moreover the conditions (12) or (14) are satisfied then  $\{U_V(t); t \geq 0\}$  is a compact semigroup.

*Proof:* (i) Let  $P_{\Omega_M} : L^1(\Omega; \mu) \rightarrow L^1(\Omega_M; \mu)$  be the restriction operator. Note that

$$P_{\Omega_M}(\lambda - T)^{-1} = P_{\Omega_M} \int_0^{+\infty} e^{-\lambda t} U(t) dt = \lim_{\varepsilon \rightarrow 0} P_{\Omega_M} \int_{\varepsilon}^{\varepsilon^{-1}} e^{-\lambda t} U(t) dt$$

where the convergence holds in operator norm. Let us show (16) or equivalently that  $P_{\Omega_M}(\lambda - T)^{-1}$  is weakly compact. It suffices to show that

$$P_{\Omega_M} \int_{\varepsilon}^{\varepsilon^{-1}} e^{-\lambda t} U(t) dt = \int_{\varepsilon}^{\varepsilon^{-1}} e^{-\lambda t} P_{\Omega_M} U(t) dt$$

is a weakly compact operator. This is a strong integral (not a Bochner integral) of a bounded, strongly continuous  $W(L^1(\Omega; \mu), L^1(\Omega_M; \mu))$ -valued mapping where  $W(L^1(\Omega; \mu), L^1(\Omega_M; \mu))$  is the Banach space of weakly compact operators from  $L^1(\Omega; \mu)$  into  $L^1(\Omega_M; \mu)$ . By [75] or [57]  $\int_{\varepsilon}^{\varepsilon^{-1}} e^{-\lambda t} P_{\Omega_M} U(t) dt$  is a weakly compact operator. Then the first claim is a consequence of Theorem 7.

(ii) We can choose  $\bar{t}$  as small as we want such that (13) is satisfied with  $t = \bar{t}$ . Let  $f = U_V(\bar{t})g$  with  $g \in B$  the unit ball of  $L^1(\Omega; \mu)$ . We note that (13) implies that  $\int_{\{V(x) \geq M\}} |f(x)| \mu(dx) \rightarrow 0$  as  $M \rightarrow +\infty$  uniformly in  $g \in B$ . On the other hand

$$|f| = |U_V(\bar{t})g| \leq U_V(\bar{t})|g| \leq U(\bar{t})|g|$$

so that, by (17), the restriction to  $\Omega_M$  of  $\{U_V(\bar{t})g; g \in B\}$  is relatively weakly compact by domination and then, by arguing as in the proof of Theorem 7, one sees that  $\{U_V(\bar{t})g; g \in B\}$  is a relatively weakly compact subset of  $L^1(\Omega; \mu)$ , i.e.  $U_V(t)$  is a weakly compact operator for all  $t \geq \bar{t}$  and consequently for all  $t > 0$ . Actually,  $U_V(t)$  is a compact operator for all  $t > 0$  since  $U_V(t) = U_V(\frac{t}{2})U_V(\frac{t}{2})$  and the product of two weakly compact

operators on  $L^1(\Omega; \mu)$  is a compact operator (see e.g. [2] Corollary 5.88, p. 344). ■

We give now an alternative proof of Theorem 10(ii).

**Theorem 11** *Let (16) be satisfied and let  $\{U_V(t); t \geq 0\}$  be norm continuous. Then  $U_V(t)$  is compact for all  $t > 0$ .*

*Proof:* It is a standard fact from semigroup theory (see e.g. [65] Theorem 1.25, p 41) that under the norm continuity of  $\{U_V(t); t \geq 0\}$ , the compactness of  $(\lambda - T_V)^{-1}$  is equivalent to the compactness of  $U_V(t)$  for  $t > 0$ . We can give here a different argument adapted to our particular context. We have (for large  $\lambda$  and) for any  $\bar{t} > 0$  and  $\varepsilon > 0$

$$(\lambda - T_V)^{-1} = \int_0^{+\infty} e^{-\lambda t} U_V(t) dt \geq \int_{\bar{t}}^{\bar{t}+\varepsilon} e^{-\lambda t} U_V(t) dt$$

so that, for any  $\varepsilon > 0$ ,  $\varepsilon^{-1} \int_{\bar{t}}^{\bar{t}+\varepsilon} e^{-\lambda t} U_V(t) dt$  is a weakly compact operator by domination. Letting  $\varepsilon \rightarrow 0$  and using the right continuity of  $t \rightarrow U_V(t)$  in operator norm we obtain that  $U_V(\bar{t})$  is a weakly compact operator for all  $\bar{t} > 0$  and consequently, as previously,  $U_V(t)$  is compact for all  $t > 0$ . ■

One sees then how important is the norm continuity of  $\{U_V(t); t \geq 0\}$ ; it provides us with a useful mean to translate compactness properties from the resolvent  $(\lambda - T_V)^{-1}$  to the semigroup  $U_V(t)$ . It is an interesting open problem to decide whether the norm continuity of  $\{U(t); t \geq 0\}$  implies that of  $\{U_V(t); t \geq 0\}$ . This problem is not covered by the paper [52] dealing with unbounded perturbations preserving immediate norm continuity of the semigroup. We provide here a solution to this open problem.

**Theorem 12** *Let  $V : (\Omega; \mu) \rightarrow [0, +\infty[$  and  $V_n := V \wedge n$ .*

*(i) Then for all finite  $C > 0$*

$$\sup_{t \leq C} \|e^{t(T-V_n)} f - U_V(t)f\| \leq e^C \|[V - V_n](1 - T_V)^{-1}f\|, \quad \forall f \in L_+^1(\Omega; \mu).$$

*In particular, if  $\|[V - V_n](1 - T_V)^{-1}\|_{\mathcal{L}(L^1(\Omega; \mu))} \rightarrow 0$  as  $n \rightarrow +\infty$  and if  $\{U(t); t \geq 0\}$  is norm continuous then  $\{U_V(t); t \geq 0\}$  is also norm continuous.*

*(ii) In particular, let  $(1 - T_V)^{-1}$  be an integral operator with kernel  $G_V(x, y)$ . If  $\{U(t); t \geq 0\}$  is norm continuous and if*

$$\sup_{y \in \Omega} \int_{\{V \geq n\}} G_V(x, y) V(x) \mu(dx) \rightarrow 0 \text{ as } n \rightarrow +\infty$$

*then  $\{U_V(t); t \geq 0\}$  is also norm continuous.*

*Proof:* Note first that both  $V$  and  $V_n$  are  $T_V$ -bounded so that the sequence  $\{[V - V_n](1 - T_V)^{-1}\}_n$  of bounded operators converges strongly to zero. According to the general theory  $e^{t(T-V_n)}f \rightarrow U_V(t)f$  for all  $f \in L^1(\Omega; \mu)$  uniformly in  $t \in [0, C]$ . We start with the Duhamel formula (for a positive bounded perturbation) and  $f \in L^1_+(\Omega; \mu)$

$$e^{t(T-V_n)}f = e^{t(T-V_{n+k})}f + \int_0^t e^{(t-s)(T-V_{n+k})} [V_{n+k} - V_n] e^{s(T-V_{n+k})} f ds.$$

By letting  $k \rightarrow +\infty$ ,  $V_{n+k}(x) \rightarrow V(x) - V_n(x)$  a.e. and then

$$e^{t(T-V_n)}f = U_V(t)f + \int_0^t U_V(t-s) [V - V_n] U_V(s)f ds.$$

The additivity of the norm on the positive cone shows that

$$\begin{aligned} \|e^{t(T-V_n)}f - U_V(t)f\| &= \left\| \int_0^t U_V(t-s) [V - V_n] U_V(s)f ds \right\| \\ &= \int_0^t \|U_V(t-s) [V - V_n] U_V(s)f\| ds \\ &\leq \int_0^t \|[V - V_n] U_V(s)f\| ds = \left\| \int_0^t [V - V_n] U_V(s)f ds \right\| \\ &= \left\| [V - V_n] \int_0^t U_V(s)f ds \right\| \leq \left\| [V - V_n] \int_0^C U_V(s)f ds \right\| \\ &\leq e^C \left\| [V - V_n] \int_0^C e^{-s} U_V(s)f ds \right\| \end{aligned}$$

for all  $t \leq C$  where  $C > 0$  is arbitrary. Hence

$$\sup_{t \leq C} \|e^{t(T-V_n)}f - U_V(t)f\| \leq e^C \|[V - V_n](1 - T_V)^{-1}f\|, \quad \forall f \in L^1_+(\Omega; \mu)$$

and

$$\sup_{t \leq C} \|e^{t(T-V_n)} - U_V(t)\| \leq e^C \|[V - V_n](1 - T_V)^{-1}\|.$$

Finally, if  $\{U(t); t \geq 0\}$  is norm continuous then so is  $\{e^{t(T-V_n)}; t \geq 0\}$  because  $V_n$  is a bounded perturbation [69] so that the last operator norm estimate ends the proof of (i). If  $(1 - T_V)^{-1}$  is an integral operator with kernel  $G_V(x, y)$  then an elementary calculation shows that

$$\|[V - V_n](1 - T_V)^{-1}\|_{\mathcal{L}(L^1(\Omega))} = \sup_{y \in \Omega} \int_{\{V \geq n\}} G_V(x, y) V(x) \mu(dx)$$

and this combined with (i) end the proof of (ii). ■

The category of holomorphic semigroups is also particularly interesting since the holomorphy of  $\{U(t); t \geq 0\}$  implies that of  $\{U_V(t); t \geq 0\}$  [4][41] and therefore Theorem 11 implies:

**Corollary 13** *Let (16) be satisfied. If  $\{U(t); t \geq 0\}$  is holomorphic or under the conditions of Theorem 12 (ii),  $\{U_V(t); t \geq 0\}$  is a compact semigroup.*

**Remark 14** *It is not difficult to see that  $(\lambda - T_V)^{-1}$  is compact if and only if  $\int_0^t U_V(s)ds$  is for all  $t \geq 0$  (the argument holds for general  $c_0$ -semigroups in Banach spaces). Thus  $\int_0^t U_V(s)ds$  is a compact operator on  $L^1(\Omega; \mu)$  under Assumption (16) only.*

**Remark 15** (i) *The assumption that  $(\lambda - T)^{-1}$  (resp.  $U(t)$ ) :  $L^1(\Omega; \mu) \rightarrow L^1(\Omega_M; \mu)$  is weakly compact is trivially satisfied if the sublevels sets have finite measure and  $(\lambda - T)^{-1}$  (resp.  $U(t)$ ) maps continuously  $L^1(\Omega; \mu)$  into  $L^p(\Xi; \mu)$  for some  $p > 1$  for any Borel set  $\Xi$  with finite measure, (e.g. for ultracontractive symmetric Markov semigroups); this follows from the fact that for  $p > 1$ , a bounded subset of  $L^p(\Xi; \mu)$  is equi-integrable.*

(ii) *When  $L^1(\Omega; \mu)$  is separable, such weak compactness assumptions imply that  $(\lambda - T)^{-1} : L^1(\Omega; \mu) \rightarrow L^1(\Omega_M; \mu)$  (resp.  $U(t) : L^1(\Omega; \mu) \rightarrow L^1(\Omega_M; \mu)$ ) is an integral operator with a measurable kernel (see the remark in [24] p. 508) and this clearly implies that  $(\lambda - T)^{-1} : L^1(\Omega; \mu) \rightarrow L^1(\Omega; \mu)$  (resp.  $U(t) : L^1(\Omega; \mu) \rightarrow L^1(\Omega; \mu)$ ) is an integral operator with a measurable kernel (provided that  $\cup_M \Omega_M = \Omega$ , i.e.  $V$  is finite a.e.). In particular, this is the case of ultracontractive symmetric Markov semigroups (see also [77] Corollary A.1.2).*

In the special case where  $\{U(t); t \geq 0\}$  operates on all  $L^p$  spaces, we can extend the above results to  $L^p$  spaces by interpolation. Suppose that  $\{U_p(t); t \geq 0\}$  are  $c_0$ -semigroups on  $L^p(\Omega; \mu)$  with generator  $T_p$  ( $p \geq 1$ ) such that  $U_p(t)$  and  $U_q(t)$  coincide on  $L^p(\Omega; \mu) \cap L^q(\Omega; \mu)$ . We note  $\{U(t); t \geq 0\}$  instead of  $\{U_1(t); t \geq 0\}$ . Then, as in the  $L^1$  case, we define the semigroups  $\{U_{pV}(t); t \geq 0\}$  with generator  $T_{pV}$  and  $\{U_{pV}(t); t \geq 0\}$  are strongly continuous if and only if  $\{U_V(t); t \geq 0\}$  is ([83] Proposition 3.1). (We point out that in  $T_{pV}$  and  $U_{pV}(t)$ , the subscript  $pV$  is *not* the product of  $p$  and  $V$  !) Then using the compactness interpolation theorem for  $\sigma$ -finite measures (see e.g. [18] Theorem 1.6.1, p. 35) we obtain immediately:

**Theorem 16** *Let  $\mu$  be  $\sigma$ -finite. If (16) or (17) is satisfied then  $T_{pV}$  is resolvent compact in  $L^p(\Omega; \mu)$ . Moreover if (12) or (14) are satisfied (e.g. if*



$\{U(t); t \geq 0\}$  is holomorphic) or under the conditions of Theorem 12 (ii), the semigroups  $\{U_{pV}(t); t \geq 0\}$  are compact in  $L^p(\Omega; \mu)$ .

If  $\{U_2(t); t \geq 0\}$  is self-adjoint, we obtain more precise results:

**Theorem 17** *Let (16) be satisfied. If  $\{U_2(t); t \geq 0\}$  is self-adjoint then the semigroup  $\{U_{pV}(t); t \geq 0\}$  is compact in  $L^p(\Omega; \mu)$  for  $p > 1$ .*

*Proof:* By interpolation (see e.g. [18] Theorem 1.6.1, p. 35), the generator of the self-adjoint semigroup  $\{U_{2V}(t); t \geq 0\}$  is resolvent compact and then the semigroup itself is compact for  $t > 0$ . Actually this result can also be obtained as follows: being self-adjoint,  $]0, +\infty[ \ni t \rightarrow U_{2V}(t) \in \mathcal{L}(L^2(\Omega; \mu))$  is continuous in operator norm topology and then we can argue as in the proof of Theorem 11 by using compactness results by domination in  $L^p$  spaces when  $p > 1$  (Dodds–Fremlin’s Theorem); see e.g. [2] Theorem 5.20, p. 286. The case  $p \neq 2$  ( $p > 1$ ) follows by interpolation again. ■

**Remark 18** *Note that if  $\{U_2(t); t \geq 0\}$  is self-adjoint then, under (16) only, the semigroup  $\{U_V(t); t \geq 0\}$  is not a priori compact on  $L^1(\Omega; \mu)$ .*

We end this section by showing that the basic assumption (17) is stable by subordination. We recall first some notions on subordinate semigroups. Let  $f \in C^\infty((0, +\infty))$  be a Bernstein function, i.e.

$$f \geq 0, \quad (-1)^k \frac{d^k f(x)}{dx^k} \leq 0 \quad \forall k \in \mathbb{N}.$$

It is characterized by the representation  $e^{-tf(x)} = \int_0^{+\infty} e^{-xs} \eta_t(ds)$  ( $t > 0$ ) where  $(\eta_t)_{t \geq 0}$  is a convolution semigroup of measures on  $[0, +\infty)$  (see e.g. [42] Theorem 3.9.7, p. 177). Let  $\{U(t); t \geq 0\}$  be a contraction semigroup. We can define (see [42] Chapter 4 for the details) the so-called subordinate semigroup  $\{U^f(t); t \geq 0\}$  acting as

$$\varphi \in L^1(\mathbb{R}^N) \rightarrow U^f(t)\varphi = \int_0^{+\infty} (U(s)\varphi)\eta_t(ds) \in L^1(\mathbb{R}^N).$$

**Theorem 19** *Let  $\{U(t); t \geq 0\}$  be a positive contraction semigroup satisfying (17). Let  $f$  be a Bernstein function such that  $f(x) \rightarrow +\infty$  as  $x \rightarrow +\infty$ . Then the subordinate semigroup  $\{U^f(t); t \geq 0\}$  satisfies also (17).*

*Proof:* Note first that  $f(x) \rightarrow +\infty$  as  $x \rightarrow +\infty$  (or equivalently: for all  $t > 0$ ,  $e^{-tf(x)} \rightarrow 0$  as  $x \rightarrow +\infty$ ) amounts to  $\eta_t(\{0\}) = 0 \forall t > 0$ . This implies that

$$\left\| \int_{\varepsilon}^{\varepsilon^{-1}} U(s) \eta_t(ds) - U^f(t) \right\| \leq \eta_t([0, \varepsilon]) + \eta_t(] \varepsilon^{-1}, +\infty[) \rightarrow 0 \text{ as } \varepsilon \rightarrow 0,$$

so that

$$\left\| \int_{\varepsilon}^{\varepsilon^{-1}} P_{\Omega_M} U(s) \eta_t(ds) - P_{\Omega_M} U^f(t) \right\| \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$

It suffices then to show that  $\int_{\varepsilon}^{\varepsilon^{-1}} P_{\Omega_M} U(s) \eta_t(ds)$  is a weakly compact operator. By assumption,  $\forall s > 0$ ,  $P_{\Omega_M} U(s)$  is a weakly compact operator. Moreover

$$s > 0 \rightarrow P_{\Omega_M} U(s) \in \mathcal{L}(L^1(\mathbb{R}^N), L^1(\Omega_M))$$

is strongly continuous and bounded. It follows from [75] or [57] that the strong integral  $\int_{\varepsilon}^{\varepsilon^{-1}} P_{\Omega_M} U(s) \eta_t(ds)$  is a weakly compact operator. ■

### 3 Applications to convolution semigroups

This section is devoted to specific results on convolution semigroups in euclidean spaces because of their importance in applications. More general situations are dealt with in Section 4. Let  $\Omega \subset \mathbb{R}^N$  be a Borel subset. We say that  $\Omega$  is “thin at infinity” if

$$|\Omega \cap B(z; 1)| \rightarrow 0 \text{ as } z \rightarrow \infty \quad (18)$$

where  $B(z; 1)$  is the ball with radius 1 centered at  $z \in \mathbb{R}^N$  and  $||$  refers to Lebesgue measure. We start with a basic result.

**Lemma 20** *Let  $H: \varphi \in L^1(\mathbb{R}^N) \rightarrow \int h(x-y)\varphi(y)dy \in L^1(\mathbb{R}^N)$  be a convolution operator with  $h \in L^1_+(\mathbb{R}^N)$ . Then  $H: \varphi \in L^1(\mathbb{R}^N) \rightarrow L^1(\Omega)$  is compact if and only if*

$$\sup_{y \in \mathbb{R}^N} \int_{\Omega \cap \{|x| \geq c\}} h(x-y) dx \rightarrow 0 \text{ as } c \rightarrow +\infty. \quad (19)$$

*Moreover (19) is satisfied if  $\Omega$  is “thin at infinity”.*

*Proof:* We note first that the continuity of  $y \in \mathbb{R}^N \rightarrow h^y(\cdot) \in L^1(\mathbb{R}^N)$  (where  $h^y(\cdot) : x \rightarrow h(x - y)$  is the translation of  $h(\cdot)$  by a vector  $y$ ) shows that  $H : L^1(\mathbb{R}^N) \rightarrow L^1(\Xi)$  is compact for any bounded Borel set  $\Xi$ . On the other hand, if  $H : \varphi \in L^1(\mathbb{R}^N) \rightarrow L^1(\Omega)$  is compact then

$$\left\| \chi_{\Omega \cap \{|x| > c\}} H \right\|_{\mathcal{L}(L^1(\mathbb{R}^N), L^1(\Omega))} \rightarrow 0 \quad \text{as } c \rightarrow +\infty$$

(we still denote by  $\chi_{\Omega \cap \{|x| > c\}}$  the multiplication operator by the indicator function  $\chi_{\Omega \cap \{|x| > c\}}$ ) because  $\left\| \chi_{\{|x| > c\}} f \right\|_{L^1(\Omega)} \rightarrow 0$  as  $c \rightarrow +\infty$  uniformly in  $f$  in a compact set of  $L^1(\Omega)$ , i.e. (19) holds. Conversely, under (19),  $H : \varphi \in L^1(\mathbb{R}^N) \rightarrow L^1(\Omega)$  is a limit in operator norm (as  $c \rightarrow +\infty$ ) of  $\chi_{\Omega \cap \{|x| \leq c\}} H$  which is compact since  $\Omega \cap \{|x| \leq c\}$  is bounded. Let us show now that (19) is satisfied if  $\Omega$  is “thin at infinity”. To show (19) it suffices that

$$\lim_{|y| \rightarrow +\infty} \int_{\Omega} h(x - y) dx = 0. \quad (20)$$

Indeed, let  $\varepsilon > 0$  be arbitrary and let  $D > 0$  be such that

$$\int_{\Omega} h(x - y) dx \leq \varepsilon \quad \text{for all } |y| > D.$$

It suffices to show that for any  $D > 0$

$$\sup_{|y| \leq D} \int_{\Omega \cap \{|x| \geq c\}} h(x - y) dx \rightarrow 0 \quad \text{as } c \rightarrow +\infty$$

i.e.

$$\sup_{|y| \leq D} \int_{\Omega \cap \{|x| \geq c\}} h^y(x) dx \rightarrow 0 \quad \text{as } c \rightarrow +\infty. \quad (21)$$

Since  $y \in \mathbb{R}^N \rightarrow h^y(\cdot) \in L^1(\mathbb{R}^N)$  is continuous then

$$\{h^y(\cdot); |z| \leq D\} \text{ is compact subset of } L^1(\mathbb{R}^N)$$

and consequently  $\{h^y(\cdot); |z| \leq D\}$  is an equi-integrable subset of  $L^1(\mathbb{R}^N)$  so that (21) is true. It suffices now to show that (20) is satisfied if  $\Omega$  is “thin at infinity”. We observe first that (18) is actually equivalent to

$$\forall R \geq 1, |\Omega \cap B(y; R)| \rightarrow 0 \quad \text{as } |y| \rightarrow \infty \quad (22)$$

where  $B(y; R)$  is the ball with radius  $R$  centered at  $y \in \mathbb{R}^N$ . It suffices to observe that  $|\Omega \cap B(y; R)| \leq \sum_{i=1}^{J_R} |\Omega \cap B(y_i; 1)|$  where we have covered

$B(y; R)$  by a finite number  $J_R$  (depending on  $R$  only) of balls  $B(y_i; 1)$  with radius 1. We write

$$\begin{aligned} \int_{\Omega} h(x-y)dx &= \int_{\Omega-y} h(z)dz = \int_{(\Omega-y) \cap B(0,R)} h(z)dz + \int_{(\Omega-y) \cap B(0,R)^c} h(z)dz \\ &\leq \int_{(\Omega-y) \cap B(0,R)} h(z)dz + \int_{B(0,R)^c} h(z)dz \end{aligned}$$

where  $B(0; R)^c$  is the exterior of the ball  $B(0; R)$ . The invariance of Lebesgue measure by translation yields

$$|(\Omega - y) \cap B(0; R)| = |\Omega \cap B(y; R)|. \quad (23)$$

Finally, for any  $\varepsilon > 0$  we choose  $R$  large enough so that  $\int_{B(0; R)^c} h(z)dz < \varepsilon$  and then  $\int_{(\Omega-y) \cap B(0; R)} h(z)dz \rightarrow 0$  as  $|y| \rightarrow +\infty$  by (22) and (23). ■

We consider now the convolution semigroups

$$U_p(t) : f \in L^p(\mathbb{R}^N) \rightarrow \int f(x-y)m_t(dy) \in L^p(\mathbb{R}^N)$$

defined in the Introduction. Such convolution semigroups, related to Lévy processes, cover many examples of practical interest such as Gaussian semigroups,  $\alpha$ -stable semigroups, relativistic Schrödinger semigroups, relativistic  $\alpha$ -stable semigroup etc. (see [42] Chapter 3). The semigroups  $\{U_p(t); t \geq 0\}_{t \geq 0}$  are strongly continuous positive contractions on  $L^p(\mathbb{R}^N)$  for  $1 \leq p < +\infty$  with generator  $T_p$ . We recall that

$$(\lambda - T)^{-1}f = \int f(x-y)m^\lambda(dy)$$

where

$$\widehat{m^\lambda}(\zeta) = \int_0^{+\infty} e^{-\lambda t} \widehat{m_t}(\zeta) dt = \frac{1}{\lambda + F(\zeta)}.$$

We make the assumption that

$$\exists G_\lambda \in L^1_+(\mathbb{R}^N) \text{ such that } \widehat{G_\lambda}(\xi) = \frac{1}{\lambda + F(\xi)}. \quad (24)$$

Note that (24) is satisfied if, for all  $t > 0$ ,  $m_t$  is a function, i.e.  $U(t)$  is a convolution operator with a kernel  $p_t(\cdot) \in L^1_+(\mathbb{R}^N)$ . As a consequence of Lemma 20 we have:

**Theorem 21** *Let (24) be satisfied. If the sublevel sets  $\Omega_M$  are “thin at infinity” then  $T_V$  is resolvent compact.*

Since  $\{U_2(t); t \geq 0\}$  is self-adjoint for real characteristic exponent then Theorem 17 implies:

**Corollary 22** *We assume that the characteristic exponent is real. Let (24) be satisfied and  $\Omega_M$  be “thin at infinity”. Then  $\{U_{pV}(t); t \geq 0\}$  are compact semigroups for all  $p > 1$ .*

The fact that  $m^\lambda$  is a function if  $e^{-tF(\zeta)} \in L^1(\mathbb{R}^N)$  ( $t > 0$ ) implies:

**Corollary 23** *We assume that  $e^{-tF(\zeta)} \in L^1(\mathbb{R}^N)$  ( $t > 0$ ). Then  $T_V$  is resolvent compact if the sublevel sets  $\Omega_M$  are “thin at infinity”.*

Thus, the semigroups generated by (non-selfadjoint) second order elliptic operators with constant coefficients, the  $\alpha$ -stable semigroup, the relativistic schrodinger operator or more generally the relativistic  $\alpha$ -stable semigroup are all covered by Corollary 23. These examples are also covered by the following theorem with a stronger conclusion.

**Theorem 24** *Let  $f$  be a Bernstein function such that  $f(x) \rightarrow +\infty$  as  $x \rightarrow +\infty$  and let  $\{U^f(t); t \geq 0\}$  be the corresponding subordinate Brownian semigroup on  $L^1(\mathbb{R}^N)$ . Then  $\{U_V^f(t); t \geq 0\}$  is a compact semigroup if the sublevel sets  $\Omega_M$  are “thin at infinity”.*

*Proof:* Let  $\{U(t); t \geq 0\}$  be the heat semigroup. Since (17) be clearly satisfied, then Theorem 19 shows that (17) (and consequently (16)) is satisfied by its subordinate semigroup  $\{U^f(t); t \geq 0\}$ . The latter being holomorphic, Theorem 11 shows that  $\{U_V^f(t); t \geq 0\}$  is a compact semigroup. ■

**Remark 25** *We note that the geometric  $\alpha$ -stable semigroup corresponding to  $F(\zeta) = \ln(1 + |\zeta|^\alpha)$  where  $0 < \alpha \leq 2$  satisfies  $e^{-tF(\zeta)} \in L^1(\mathbb{R}^N)$  for  $t > \frac{N}{\alpha}$  only so that Corollary 23 does not apply a priori. However, by subordination to heat semigroup, this case is covered by Theorem 24.*

**Remark 26** (i) *Note that we have now a stronger version of Theorem 1 since, under (3),  $\Delta - V$  generates a (holomorphic) compact semigroup in  $L^1(\mathbb{R}^N)$ . In particular, this is the case in  $L^1(\mathbb{R}^2)$  for the potential (1). Indeed, in this case,  $\Omega_M = \{(x_1, x_2); |x_2| \leq \frac{M}{|x_1|}\}$ . It suffices to restrict ourselves to*

$$\Omega_M^+ := \Omega_M \cap \{(x_1, x_2); x_1 > 0, x_2 > 0\} = \left\{ (x_1, x_2); x_2 \leq \frac{M}{x_1} \right\}$$

and to consider for instance the case where we move the ball  $B(z; 1)$  (centered at  $z = (z_1, z_2)$  with  $z_1 > 0$ ) by letting  $z_1 \rightarrow +\infty$ . The set  $B(z; 1) \cap \Omega_M^+$  is included in  $\{(x_1, x_2); z_1 - 1 \leq x_1 \leq z_1 + 1\} \cap \Omega_M^+$  whose Lebesgue measure is equal to

$$\int_{z_1-1}^{z_1+1} \frac{M}{x_1} dx_1 = M \ln\left(\frac{z_1+1}{z_1-1}\right) \rightarrow 0 \quad \text{as } z_1 \rightarrow +\infty.$$

(ii) Similar arguments apply to  $\Delta - (x^2 y^2 + x^2 z^2 + y^2 z^2)$  in  $L^1(\mathbb{R}^3)$ ; see Section 7 for much more examples arising in the study of weighted Laplacians.

**Remark 27** We note that we can easily deal with non translation-invariant positive semigroups  $\{U(t); t \geq 0\}$  on  $L^1(\Omega; dx)$  where  $\Omega \subset \mathbb{R}^N$  by using domination arguments provided that  $\{U(t); t \geq 0\}$  admits a kernel estimate of convolution type. More systematic results, including spectral gaps, are given in the next two sections in the more general context of  $L^1$  spaces over metric measure spaces.

## 4 Metric measure spaces and compactness

The interest of Markov processes in metric spaces (see e.g. [32] and references therein) suggests naturally to investigate the compactness (or spectral gap) problems in Lebesgue spaces over metric measure spaces. Let  $(\Omega, d)$  be a metric space and  $\mu$  be a Borel measure on  $\Omega$  which is finite on bounded Borel sets. We assume in this section and in the next one that  $L^1(\Omega; \mu)$  is separable. Let  $\{U(t); t \geq 0\}$  be a sub-stochastic  $c_0$ -semigroup on  $L^1(\Omega; \mu)$  with generator  $T$ . We show here how the existence of a metric allows us to precise further some results of Section 2.

**Theorem 28** We assume that  $(1 - T)^{-1} : L^1(\Omega; \mu) \rightarrow L^1(\Xi)$  is weakly compact for any bounded Borel set  $\Xi$ . Let  $G_1(x, y)$  be the kernel of  $(1 - T)^{-1}$ . If

$$\lim_{C \rightarrow +\infty} \sup_{y \in \Omega} \int_{\{x \in \Omega_M; d(x, x_0) \geq C\}} G_1(x, y) \mu(dx) = 0, \quad \forall M > 0 \quad (25)$$

(for some  $x_0 \in \Omega$ ) then  $T_V$  is resolvent compact.

*Proof:* Note that (25) is  $x_0$ -independent. As in Remark 15 (ii), the existence of the kernel  $G_1(x, y)$  follows from the separability of  $L^1(\Omega; \mu)$  and the weak compactness assumption, see the remark in [24] p. 508. One sees,

by domination, that  $(1 - T_V)^{-1} : L^1(\Omega; \mu) \rightarrow L^1(\Xi)$  is also weakly compact for any bounded Borel set  $\Xi$  and then  $(1 - T_V)^{-1}$  has also a kernel  $G_1^V(x, y)$ . We decompose  $(1 - T_V)^{-1}$  as

$$\begin{aligned} (1 - T_V)^{-1} &= \chi_{\Omega_M^c} (1 - T_V)^{-1} + \chi_{\{x \in \Omega_M, d(x, x_0) \geq C\}} (1 - T_V)^{-1} \\ &\quad + \chi_{\{x \in \Omega_M, d(x, x_0) < C\}} (1 - T_V)^{-1} \end{aligned}$$

where  $\Omega_M^c$  is the complement of the sublevel set  $\Omega_M$ . Since

$$\chi_{\{x \in \Omega_M, d(x, x_0) < C\}} (1 - T_V)^{-1} \leq \chi_{\{x \in \Omega_M, d(x, x_0) < C\}} (1 - T)^{-1}$$

then, by our assumption,  $\chi_{\{x \in \Omega_M, d(x, x_0) < C\}} (1 - T_V)^{-1}$  is weakly compact. Moreover, we saw in the proof of Theorem 7 that the norm of  $\chi_{\Omega_M^c} (1 - T_V)^{-1}$  goes to zero as  $M \rightarrow +\infty$ . The norm of  $\chi_{\{x \in \Omega_M, d(x, x_0) \geq C\}} (1 - T_V)^{-1}$  is less than or equal to that of  $\chi_{\{x \in \Omega_M, d(x, x_0) \geq C\}} (1 - T)^{-1}$  i.e.

$$\sup_{y \in \Omega} \int_{\{x \in \Omega_M; d(x, x_0) \geq C\}} G_1(x, y) \mu(dx).$$

Thus  $\left\| (1 - T)^{-1} - \chi_{\{x \in \Omega_M, d(x, x_0) < C\}} (1 - T_V)^{-1} \right\|_{\mathcal{L}(L^1(\Omega; \mu))}$  is arbitrarily small for  $M$  and  $C$  large enough. Hence  $(1 - T_V)^{-1}$  is weakly compact and Lemma 6 ends the proof. ■

We note that under the conditions of Theorem 12 (ii)  $\{U_V(t); t \geq 0\}$  is norm continuous and then Theorem 28 implies the compactness of the semigroup  $\{U_V(t); t \geq 0\}$  (see Theorem 11). We can also derive this result differently under other conditions.

**Theorem 29** *We assume that (12) or (14) is satisfied. Let  $U(t) : L^1(\Omega; \mu) \rightarrow L^1(\Xi)$  be weakly compact for any bounded Borel set  $\Xi$  and  $t > 0$  and let  $p_t(x, y)$  be its kernel. If for all  $t > 0$*

$$\lim_{C \rightarrow +\infty} \sup_{y \in \Omega} \int_{\{x \in \Omega_M; d(x, x_0) \geq C\}} p_t(x, y) \mu(dx) = 0 \quad (26)$$

(for some  $x_0 \in \Omega$ ) then  $\{U_V(t); t \geq 0\}$  is a compact semigroup.

*Proof:* Note that (26) is  $x_0$ -independent. Arguing as in the previous proof, one sees that  $U(t)$  and  $U_V(t)$  have kernels  $p_t(x, y)$  and  $p_t^V(x, y)$ . We decompose  $U_V(t)$  as

$$\begin{aligned} U_V(t) &= \chi_{\Omega_M^c} U_V(t) + \chi_{\{x \in \Omega_M, d(x, x_0) \geq C\}} U_V(t) \\ &\quad + \chi_{\{x \in \Omega_M, d(x, x_0) < C\}} U_V(t) \end{aligned} \quad (27)$$

where  $\Omega_M^c$  is the complement of the sublevel set  $\Omega_M$ . Since

$$\chi_{\{x \in \Omega_M, d(x, x_0) < C\}} U_V(t) \leq \chi_{\{x \in \Omega_M, d(x, x_0) < C\}} U(t)$$

then, by our assumption, the third operator in (27) is weakly compact. Moreover, by the weak-type estimate (13) (a consequence of (12)), the norm of  $\chi_{\Omega_M^c} U_V(t)$  goes to zero as  $M \rightarrow +\infty$ . Finally, the norm of  $\chi_{\{x \in \Omega_M, d(x, x_0) \geq C\}} U_V(t)$  is less than or equal to that of  $\chi_{\{x \in \Omega_M, d(x, x_0) \geq C\}} U(t)$  i.e.

$$\sup_{y \in \Omega} \int_{\{x \in \Omega_M; d(x, x_0) \geq C\}} p_t(x, y) \mu(dx).$$

Thus  $\left\| U_V(t) - \chi_{\{x \in \Omega_M, d(x, x_0) < C\}} U_V(t) \right\|_{\mathcal{L}(L^1(\Omega; \mu))}$  is arbitrarily small for  $M$  and  $C$  large enough. Hence  $U_V(t)$  is weakly compact for all  $t > 0$  and finally  $\{U_V(t); t \geq 0\}$  is a compact semigroup since  $U_V(t) = U_V(\frac{t}{2})U_V(\frac{t}{2})$  by Dunford-Pettis theorem. ■

We rely now Theorem 28 and Theorem 29 on the notion of sublevels sets “thin at infinity”. We introduce first:

**Definition 30** *We say that a Borel set  $\Xi \subset \Omega$  is “thin at infinity” if there exists a point  $\bar{y} \in \Omega$  such that for all  $M > 0$*

$$\mu \{ \Xi \cap B(y; M) \} \rightarrow 0 \quad \text{as } d(y, \bar{y}) \rightarrow +\infty \quad (28)$$

where  $B(y; M)$  is the ball centered at  $y$  with radius  $M$ .

This definition is  $\bar{y}$ -independent. We give now a basic preliminary result.

**Lemma 31** *We assume that*

$$v(r) := \sup_{x \in \Omega} \mu(B(x, r)) < +\infty, \quad \forall r \geq 0. \quad (29)$$

Let

$$H : \varphi \in L^1(\Omega; \mu) \rightarrow \int_{\Omega} h(x, y) \varphi(y) \mu(dy)$$

with a kernel estimate of the form  $h(x, y) \leq f(d(x, y))$  where  $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is nonincreasing and such that (for sufficiently large  $r$ )  $r \rightarrow f(r)v(r+1)$  is nonincreasing and integrable at infinity. Then:

- (i)  $H$  is bounded operator on  $L^1(\Omega; \mu)$ .
- (ii) If a Borel set  $\Xi \subset \Omega$  is “thin at infinity” in the sense (28) then operator  $H : \varphi \in L^1(\Omega; \mu) \rightarrow L^1(\Xi; \mu)$  is weakly compact.



*Proof:* (i) By domination, it suffices to show that

$$\varphi \in L^1(\Omega; \mu) \rightarrow \int f(d(x, y))\varphi(y)\mu(dy) \in L^1(\Omega; \mu) \quad (30)$$

is a bounded operator. This holds if and only if there exists  $C > 0$  such that

$$\int f(d(x, y))\mu(dx) \leq C \quad \forall y \in \Omega.$$

We have

$$\begin{aligned} \int f(d(x, y))\mu(dx) &= \int_{\{d(x, y) < 1\}} f(d(x, y))\mu(dx) \\ &\quad + \sum_{n=1}^{\infty} \int_{\{n \leq d(x, y) < n+1\}} f(d(x, y))\mu(dx) \\ &\leq f(0)\mu(B(y, 1)) + \sum_{n=1}^{\infty} f(n) [\mu(B(y, n+1)) - \mu(B(y, n))] \\ &= [f(0) - f(1)]\mu(B(y, 1)) + [f(1) - f(2)]\mu(B(y, 2)) + \cdots \\ &= \sum_{n=0}^{\infty} [f(n) - f(n+1)]\mu(B(y, n+1)) \end{aligned} \quad (31)$$

which is finite if

$$\sum_{n=0}^{\infty} f(n)\mu(B(y, n+1)) < \infty, \quad \sum_{n=0}^{\infty} f(n+1)\mu(B(y, n+1)) < \infty$$

and then (31) is finite if

$$\sum_{n=0}^{\infty} f(n)v(n+1) < \infty, \quad \sum_{n=0}^{\infty} f(n+1)v(n+1) < \infty$$

or equivalently  $\sum_{n=0}^{\infty} f(n)v(n+1) < \infty$  (since  $v(n) \leq v(n+1)$ ) which follows from the condition  $\int_1^{+\infty} f(r)v(r+1)dr < \infty$  because  $r \rightarrow f(r)v(r+1)$  is nonincreasing.

(ii) We decompose the integral operator (30) by decomposing its kernel as

$$f(d(x, y)) = 1_{\Xi_c}(x)f(d(x, y)) + 1_{\widetilde{\Xi}_c}(x)f(d(x, y))$$

where

$$\Xi_c := \Xi \cap \{x; d(x, \bar{y}) \geq c\} \quad \text{and} \quad \widetilde{\Xi}_c := \Xi \cap \{x; d(x, \bar{y}) < c\}$$

since  $x \in \Xi$ . Note that  $f(d(x, y)) \leq f(0)$  so that

$$\varphi \in L^1(\Omega; \mu) \rightarrow \int 1_{\tilde{\Xi}_c}(x) f(d(x, y)) \varphi(y) u(dy) \in L^\infty(\tilde{\Xi}_c; \mu)$$

and (since  $\mu \left\{ \tilde{\Xi}_c \right\}$  is finite) the imbedding of  $L^\infty(\tilde{\Xi}_c; \mu)$  into  $L^1(\tilde{\Xi}_c; \mu)$  is weakly compact because a bounded subset of  $L^\infty(\tilde{\Xi}_c; \mu)$  is equi-integrable. It suffices to show that the norm of the second part goes to zero as  $c \rightarrow +\infty$ , i.e.

$$\sup_{y \in \Omega} \int_{\Xi \cap \{d(x, \bar{y}) \geq c\}} f(d(x, y)) \mu(dx) \rightarrow 0 \text{ as } c \rightarrow +\infty.$$

Consider first the integral

$$\begin{aligned} & \int_{\Xi \cap \{d(x, \bar{y}) \geq c\}} f(d(x, y)) \mu(dx) \\ &= \sum_{n=0}^{\infty} \int_{\{n \leq d(x, y) < n+1\} \cap \Xi \cap \{d(x, \bar{y}) \geq c\}} f(d(x, y)) \mu(dx) \\ &\leq \sum_{n=0}^{\infty} f(n) \mu[\{n \leq d(x, y) < n+1\} \cap \Xi \cap \{d(x, \bar{y}) \geq c\}]. \end{aligned}$$

We note that

$$\begin{aligned} & \sum_{n=m}^{\infty} f(n) \mu[\{n \leq d(x, y) < n+1\} \cap \Xi \cap \{d(x, \bar{y}) \geq c\}] \\ &\leq \sum_{n=m}^{\infty} f(n) \mu[\{n \leq d(x, y) < n+1\}] \\ &= \sum_{n=m}^{\infty} f(n) [\mu(B(y, n+1)) - \mu(B(y, n))] \\ &\leq \sum_{n=m}^{\infty} f(n) [\mu(B(y, n+1)) + \mu(B(y, n))] \\ &\leq c \sum_{n=m}^{\infty} f(n) [v(n+1) + v(n)] \end{aligned}$$

so that, for any  $\varepsilon > 0$  there exists an integer  $\bar{m}$  such that

$$\sum_{n=\bar{m}}^{\infty} f(n) \mu[\{n \leq d(x, y) < n+1\} \cap \Xi \cap \{d(x, \bar{y}) \geq c\}] \leq \varepsilon \text{ uniformly in } y \in \Omega.$$

It suffices to show that

$$\sum_{n=0}^{\overline{m}} f(n) \mu [\{n \leq d(x, y) < n+1\} \cap \Xi \cap \{d(x, \overline{y}) \geq c\}] \rightarrow 0 \text{ as } c \rightarrow +\infty$$

uniformly in  $y \in \Omega$ , or equivalently for any  $n \leq \overline{m}$

$$\mu [\{n \leq d(x, y) < n+1\} \cap \Xi \cap \{d(x, \overline{y}) \geq c\}] \rightarrow 0 \text{ as } c \rightarrow +\infty \quad (32)$$

uniformly in  $y \in \Omega$ . The inequality

$$d(y, \overline{y}) \geq |d(x, \overline{y}) - d(x, y)| \geq c - (n+1)$$

for  $c > (n+1)$  shows that either the set

$$\{x; n \leq d(x, y) < n+1\} \cap \{x; d(x, \overline{y}) \geq c\}$$

is *empty* (and then  $\mu [\{n \leq d(x, y) < n+1\} \cap \Xi \cap \{d(x, \overline{y}) \geq c\}] = 0$ ) or  $d(y, \overline{y}) \geq c - (n+1)$ . On the other hand, by assumption, for any  $n$

$$\mu [\{x; d(x, y) < n+1\} \cap \Xi] \rightarrow 0 \text{ as } d(y, \overline{y}) \rightarrow \infty$$

and then (32) follows. ■

As a consequence of Theorem 7, Theorem 10 (ii) and Lemma 31 we have:

**Theorem 32** *Let  $(\Omega, d, \mu)$  be a measure metric space. Let  $\{U(t); t \geq 0\}$  be a sub-stochastic  $c_0$ -semigroup on  $L^1(\Omega; \mu)$  with generator  $T$  and let (29) be satisfied.*

(i) *We assume that  $(1-T)^{-1}$  is an integral operator with a kernel  $G(x, y)$  satisfying an estimate of the form  $G(x, y) \leq f(d(x, y))$  where  $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is nonincreasing and such that (for large  $r$ )  $r \rightarrow f(r)v(r+1)$  is nonincreasing and integrable at infinity. If the sublevel sets  $\Omega_M$  are “thin at infinity” in the sense (28) then  $T_V$  is resolvent compact.*

(ii) *Let  $L^1(\Omega; \mu)$  be separable and let (12) or (14) be satisfied. We assume that for each  $t > 0$ ,  $U(t)$  is an integral operator with a kernel  $p_t(., .)$  satisfying an estimate of the form  $p_t(., .) \leq f_t(d(x, y))$  where  $f_t : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is nonincreasing and such that (for large  $r$ )  $r \rightarrow f_t(r)v(r+1)$  is nonincreasing and integrable at infinity. If the sublevel sets  $\Omega_M$  are “thin at infinity” in the sense (28) then  $\{U_V(t); t \geq 0\}$  is a compact semigroup.*

Note that under the conditions of Theorem 12 (ii) the conclusion of Theorem 32 (ii) follows also from Theorem 32 (i). Note also that if we

consider the different examples of kernel estimates (5)(6)(7) arising in the theory of Markov process

$$f_t(r) := \frac{C}{t^\gamma} \exp(-\frac{r^2}{Ct}), \quad \frac{C}{t^{\frac{\alpha}{\beta}}} \exp(-\frac{r^{\frac{\beta}{\beta-1}}}{C^{\frac{\beta}{\beta-1}} t^{\frac{\beta}{\beta-1}}}) \text{ or } \frac{C}{t^{\frac{\alpha}{\beta}}} (1 + \frac{r}{t^{\frac{1}{\beta}}})^{-(\alpha+\beta)},$$

one sees which volume growth  $r \rightarrow v(r)$  is compatible with each  $f_t(\cdot)$  in order to meet the conditions in Theorem 32. .

## 5 Metric measure spaces and spectral gaps

We recall first that the spectral bound  $s(A) := \sup \{\operatorname{Re} \lambda; \lambda \in \sigma(A)\}$  of the generator  $A$  of a positive semigroup on a Banach lattice  $X$  belongs to  $\sigma(A)$  (see e.g. [65] Theorem 1.1, p. 292) and this spectral bound coincides with the type of the semigroup when  $X$  is an  $L^p(\mu)$  space (see [86]). Moreover, a  $c_0$ -semigroup  $\{S(t); t \geq 0\}$  in a Banach space has an *essential* type  $\omega_{ess}$  such that  $-\infty \leq \omega_{ess} \leq \omega$  (where  $\omega$  is the type of  $\{S(t); t \geq 0\}$ ) and

$$r_{ess}(S(t)) = e^{\omega_{ess} t}, \quad t > 0$$

where  $r_{ess}$  refers to the essential spectral radius (see e.g. [65] p. 73-74). Thus  $\lambda_V \in \sigma(T_V)$  where  $\lambda_V$  is the spectral bound of  $T_V$  and  $\lambda_V$  is also the type of  $\{U_V(t); t \geq 0\}$ . Note that  $\lambda_V \leq 0$  by the contraction of the semigroup. A spectral gap for a generator  $T_V$  refers to

$$s_{ess}(T_V) < s(T_V) \tag{33}$$

where

$$s_{ess}(T_V) := \sup \{\operatorname{Re} \lambda; \lambda \in \sigma_{ess}(T_V)\}$$

is the essential spectral bound. We give first a *weak* spectral gap result for generators.

**Theorem 33** *Let  $(\Omega, \mu, d)$  be a metric measure space. Let  $(1 - T)^{-1} : L^1(\Omega) \rightarrow L^1(\Xi)$  be weakly compact for any bounded Borel set  $\Xi$ . We assume that the kernel  $G_1(x, y)$  of  $(1 - T)^{-1}$  satisfies the estimate*

$$\sup_{M > 0} \lim_{C \rightarrow +\infty} \sup_{y \in \Omega} \int_{\{x \in \Omega_M; d(x, x_0) \geq C\}} G_1(x, y) \mu(dx) < \frac{1}{1 - \lambda_V} \tag{34}$$

(for some  $x_0 \in \Omega$ ). Then the peripheral spectrum of  $T_V$  consists of isolated eigenvalues  $\{\lambda_k\}_k$  with finite algebraic multiplicities. Moreover, there exists a positive constant  $\varepsilon$  such that for any  $k$ ,  $\sigma(T_V) \cap D(\lambda_k, \varepsilon)$  consists of isolated eigenvalues with finite algebraic multiplicities ( $D(\lambda_k, \varepsilon)$  is the disc centered at  $\lambda_k$  with radius  $\varepsilon$ ).

*Proof:* Note that (34) is  $x_0$ -independent. As previously, the existence of the kernel  $G_1(x, y)$  follows from our local weak compactness assumption. Let  $\lambda = \lambda_V + iq \in \sigma(T_V)$  be a *peripheral* spectral value. According to a standard result (see e.g. [65] Proposition 2.5, p 67), for any  $\bar{\lambda} \in \rho(T_V)$ ,

$$r_\sigma [(\bar{\lambda} - T_V)^{-1}] = \frac{1}{\text{dist}(\bar{\lambda}, \sigma(T_V))}.$$

In particular the choice of  $\bar{\lambda} = 1 + iq$  leads to

$$r_\sigma [(1 + iq - T_V)^{-1}] = \frac{1}{\text{dist}(1 + iq, \sigma(T_V))} = \frac{1}{1 - \lambda_V}$$

because  $\lambda_V + iq \in \sigma(T_V)$ . We note the standard *domination*

$$\begin{aligned} |(1 + iq - T_V)^{-1} f| &= \left| \int_0^{+\infty} e^{-(1+iq)t} e^{-tT_V} f dt \right| \\ &\leq \int_0^{+\infty} e^{-t} e^{-tT_V} |f| dt = (1 - T_V)^{-1} |f|. \end{aligned}$$

We decompose  $(1 + iq - T_V)^{-1}$  as

$$\begin{aligned} (1 + iq - T_V)^{-1} &= \chi_{\Omega_M^c} (1 + iq - T_V)^{-1} + \chi_{\{x \in \Omega_M, d(x, x_0) \geq C\}} (1 + iq - T_V)^{-1} \\ &\quad + \chi_{\{x \in \Omega_M, d(x, x_0) < C\}} (1 + iq - T_V)^{-1} \end{aligned} \quad (35)$$

where  $\Omega_M^c$  is the complement of the sublevel set  $\Omega_M$ . Since

$$\chi_{\{x \in \Omega_M, d(x, x_0) < C\}} (1 + iq - T_V)^{-1}$$

is dominated by  $\chi_{\{x \in \Omega_M, d(x, x_0) < C\}} (1 - T_V)^{-1}$  which is itself dominated by

$$\chi_{\{x \in \Omega_M, d(x, x_0) < C\}} (1 - T)^{-1}$$

then, by our assumption, the third operator in (35) is weakly compact. Moreover, we saw in the proof of Theorem 7 that the norm of  $\chi_{\Omega_M^c} (1 - T_V)^{-1}$  goes to zero as  $M \rightarrow +\infty$  so that, by domination, the norm of  $\chi_{\Omega_M^c} (1 + iq - T_V)^{-1}$  goes to zero as  $M \rightarrow +\infty$  as well. Finally, the norm of  $\chi_{\{x \in \Omega_M, d(x, x_0) \geq C\}} (1 + iq - T_V)^{-1}$  is less than or equal to that of  $\chi_{\{x \in \Omega_M, d(x, x_0) \geq C\}} (1 - T_V)^{-1}$  which is itself less than or equal to that of  $\chi_{\{x \in \Omega_M, d(x, x_0) \geq C\}} (1 - T)^{-1}$  i.e.

$$\sup_{y \in \Omega} \int_{\{x \in \Omega_M; d(x, x_0) \geq C\}} G_1(x, y) \mu(dx).$$

It follows that for  $M$  and  $C$  large enough the norm of  $\chi_{\Omega_M^c}(1+iq-T_V)^{-1} + \chi_{\{x \in \Omega_M, d(x, x_0) \geq C\}}(1+iq-T_V)^{-1}$  is less than  $\frac{1}{1-\lambda_V}$ . In  $L^1$  spaces, the essential spectrum is stable by weakly compact perturbations (see e.g. [49]) so that

$$\begin{aligned} r_{ess} [(1+iq-T_V)^{-1}] &= r_{ess} \left[ (\chi_{\Omega_M^c} + \chi_{\{x \in \Omega_M, d(x, x_0) \geq C\}})(1+iq-T_V)^{-1} \right] \\ &\leq \left\| (\chi_{\Omega_M^c} + \chi_{\{x \in \Omega_M, d(x, x_0) \geq C\}})(1+iq-T_V)^{-1} \right\| \\ &< \frac{1}{1-\lambda_V} = r_{\sigma} [(1+iq-T_V)^{-1}]. \end{aligned}$$

Since  $\lambda$  is an isolated eigenvalue of  $T_V$  with finite algebraic multiplicity if and only if  $\frac{1}{1+iq-\lambda}$  is an isolated eigenvalue of  $(1+iq-T_V)^{-1}$  with finite algebraic multiplicity, then any spectral value  $\lambda$  of  $T_V$  such that

$$\frac{1}{|1+iq-\lambda|} > r_{ess} [(1+iq-T_V)^{-1}]$$

is an isolated eigenvalue of  $T$  with finite algebraic multiplicity. Thus, any spectral value  $\lambda$  of  $T_V$  such that

$$|1+iq-\lambda| < \frac{1}{r_{ess} [(1+iq-T_V)^{-1}]}$$

is an isolated eigenvalue of  $T$  with finite algebraic multiplicity. The fact that

$$|(1+iq) - (\lambda_V + iq)| = 1 - \lambda_V < \frac{1}{r_{ess} [(1+iq-T_V)^{-1}]}$$

shows that  $\lambda_V + iq$  is an isolated eigenvalue of  $T$  with finite algebraic multiplicity; actually there exists a whole disc  $D(\lambda_V + iq, \varepsilon)$  centered at  $\lambda_V + iq$  (with radius  $\varepsilon$  independent of  $q$ ) whose intersection with  $\sigma(T_V)$  consists at most of finitely many eigenvalues. ■

The result above shows that there exists an open neighborhood of the axis  $\lambda_V + i\mathbb{R}$  whose intersection with  $\sigma(T_V)$  consists at most of isolated eigenvalues with finite algebraic multiplicities. However, a priori this neighborhood does not contain a strip of the form  $\{\lambda_V - \alpha \leq \operatorname{Re} \lambda \leq \lambda_V\}$  since we cannot prevent the existence of a sequence of eigenvalues with real parts going to  $\lambda_V$  and imaginary parts going to infinity, i.e. (33) does not occur a priori. We derive now a stronger result when  $\{U_V(t); t \geq 0\}$  is norm continuous.

**Theorem 34** *Let the conditions of Theorem 33 be satisfied. We assume that  $\{U_V(t); t \geq 0\}$  is norm continuous (e.g. the conditions of Theorem 12 (ii) are satisfied). Then  $\{U_V(t); t \geq 0\}$  has a spectral gap, i.e.  $\omega_{ess} < \lambda_V$  where  $\omega_{ess}$  is the essential type of  $\{U_V(t); t \geq 0\}$ .*

*Proof:* Let us show first that (33) holds. By the norm continuity of  $\{U_V(t); t \geq 0\}$ , we have a *Bochner* integral

$$(\lambda - T_V)^{-1} = \int_0^{+\infty} e^{-\lambda t} U_V(t) dt \quad (\operatorname{Re} \lambda > \lambda_V)$$

(instead of simply a strong integral) so that Riemann-Lebesgue Lemma holds

$$\|(\lambda - T_V)^{-1}\| \rightarrow 0 \text{ as } |\operatorname{Im} \lambda| \rightarrow \infty.$$

Suppose now that there exists a sequence of eigenvalues  $\nu_k = \alpha_k + i\beta_k$  such that  $\alpha_k \rightarrow \lambda_V$  and  $|\beta_k| \rightarrow \infty$  with normalized eigenvectors  $x_k$ . Then  $T_V x_k = (\alpha_k + i\beta_k)x_k$  so that

$$(1 + i\beta_k - T_V)x_k = (1 - \alpha_k)x_k$$

and then

$$\begin{aligned} 1 &= \|x_k\| = |(1 - \alpha_k)| \|(1 + i\beta_k - T_V)^{-1} x_k\| \\ &\leq |(1 - \alpha_k)| \|(1 + i\beta_k - T_V)^{-1}\| \end{aligned}$$

which is impossible if  $|\beta_k| \rightarrow \infty$ . This shows (33), i.e. that there exists some  $\alpha > 0$  such that  $\sigma(T_V) \cap \{\lambda_V - \alpha \leq \operatorname{Re} \lambda \leq \lambda_V\}$  consists of isolated eigenvalues with finite algebraic multiplicities. Actually the same reasoning shows that this strip cannot contain eigenvalues with imaginary parts as large as we want and finally this strip contains only finitely many eigenvalues  $\{\nu_1, \dots, \nu_J\}$ . Let  $P$  be the (finite dimensional) spectral projection corresponding to this finite set of eigenvalues. Note that this projection commutes with  $U_V(t)$ . We denote by  $Y$  its *finite dimensional* range. We decompose  $L^1(\Omega)$  as

$$L^1(\Omega) = X \oplus Y$$

where  $X = (I - P)(L^1(\Omega))$ . Then

$$\sigma(T_V) = \{\nu_1, \dots, \nu_J\} \cup \sigma(T_{V|X})$$

where  $T_{V|X}$  is the restriction of  $T_V$  to  $X$  (with domain  $D(T_V) \cap X$ ) and

$$\sigma(T_{V|X}) = \sigma(T_V) \cap \{\operatorname{Re} \lambda < \lambda_V - \alpha\}.$$

We decompose then  $U_V(t)$  as

$$U_V(t) = U_V(t)P + U_V(t)(I - P).$$

It follows that

$$\sigma_{ess}(U_V(t)) = \sigma_{ess}(U_V(t)(I - P)) \subset \sigma(U_V(t)(I - P))$$

where  $\{U_V(t)(I - P); t \geq 0\}$  is identified to the semigroup on  $X$  with generator  $T_{V|X}$ . Thus

$$r_{ess}(U_V(t)) \leq r_\sigma(U_V(t)(I - P)).$$

Since  $\{U_V(t)(I - P); t \geq 0\}$  is also norm continuous then the spectral mapping theorem

$$\sigma(U_V(t)(I - P)) - \{0\} = e^{t\sigma(T_{V|X})}$$

holds (see e.g. [65] p 87) so that  $r_\sigma(U_V(t)(I - P)) \leq e^{(\lambda_V - \alpha)t}$  and finally we get  $\omega_{ess} < \lambda_V$ . ■

We give now a second (*quantitative*) approach to spectral gaps for perturbed semigroups relying on the weak type estimate (13).

**Theorem 35** *Let  $(\Omega, \mu, d)$  be a metric measure space. We assume that (12) or (14) is satisfied (e.g.  $\{U(t); t \geq 0\}$  is holomorphic). Let  $t > 0$  be fixed and let  $U(t) : L^1(\Omega) \rightarrow L^1(\Xi)$  be weakly compact for any bounded Borel set  $\Xi$ . We assume that the kernel  $p_t(x, y)$  of  $U(t)$  satisfies the estimate*

$$\sup_{M > 0} \lim_{C \rightarrow +\infty} \sup_{y \in \Omega} \int_{\{x \in \Omega_M; d(x, x_0) \geq C\}} p_t(x, y) \mu(dx) < e^{\lambda_V t} \quad (36)$$

(for some  $x_0 \in \Omega$ ). Then  $\omega_{ess} < \lambda_V$ ; (more precisely,  $\omega_{ess} \leq \beta$  where  $e^{\beta t}$  is the left hand side of (36)). In particular, for any  $\varepsilon < \lambda_V - \omega_{ess}$ ,  $\{\lambda; \operatorname{Re} \lambda > \omega_{ess} + \varepsilon\} \cap \sigma(T_V)$  is a finite and non empty set of eigenvalues with finite algebraic multiplicities.

*Proof:* Note that (36) is  $x_0$ -independent. We decompose  $U_V(t)$  as

$$\begin{aligned} U_V(t) &= \chi_{\Omega_M^c} U_V(t) + \chi_{\{x \in \Omega_M, d(x, x_0) \geq C\}} U_V(t) \\ &\quad + \chi_{\{x \in \Omega_M, d(x, x_0) < C\}} U_V(t) \end{aligned} \quad (37)$$

where  $\Omega_M^c$  is the complement of the sublevel set  $\Omega_M$ . Since  $\chi_{\{x \in \Omega_M, d(x, x_0) < C\}} U_V(t)$  is dominated by  $\chi_{\{x \in \Omega_M, d(x, x_0) < C\}} U(t)$  then, by our assumption, the third operator in (37) is weakly compact. Moreover, by (13) (a consequence of (12)), the norm of  $\chi_{\Omega_M^c} U_V(t)$  goes to zero as  $M \rightarrow +\infty$ . Finally, the norm of  $\chi_{\{x \in \Omega_M, d(x, x_0) \geq C\}} U_V(t)$  is less than or equal to that of  $\chi_{\{x \in \Omega_M, d(x, x_0) \geq C\}} U(t)$  i.e.

$$\sup_{y \in \Omega} \int_{\{x \in \Omega_M; d(x, x_0) \geq C\}} p_t(x, y) \mu(dx).$$



For  $M$  and  $C$  large enough, the norm of  $\chi_{\Omega_M^c} U_V(t) + \chi_{\{x \in \Omega_M, d(x, x_0) \geq C\}} U_V(t)$  is less than  $e^{\lambda_V t}$ . Then the stability of the essential spectrum by weakly compact perturbations in  $L^1$  spaces (see e.g. [49]) shows that

$$\begin{aligned} e^{\omega_{ess} t} &= r_{ess}[U_V(t)] = r_{ess} \left[ (\chi_{\Omega_M^c} + \chi_{\{x \in \Omega_M, d(x, x_0) \geq C\}}) U_V(t) \right] \\ &\leq \left\| (\chi_{\Omega_M^c} + \chi_{\{x \in \Omega_M, d(x, x_0) \geq C\}}) U_V(t) \right\| < e^{\lambda_V t} \end{aligned}$$

i.e.  $\omega_{ess} < \lambda_V$ . More precisely,

$$e^{\omega_{ess} t} \leq \left\| (\chi_{\Omega_M^c} + \chi_{\{x \in \Omega_M, d(x, x_0) \geq C\}}) U_V(t) \right\| \quad \forall M, C$$

and letting  $M \rightarrow \infty$  and  $C \rightarrow \infty$  yields

$$e^{\omega_{ess} t} \leq \sup_{M > 0} \lim_{C \rightarrow +\infty} \sup_{y \in \Omega} \int_{\{x \in \Omega_M; d(x, x_0) \geq C\}} p_t(x, y) \mu(dx).$$

The remaining part follows from standard semigroup theory, see e.g. [65].

■

Actually, the proof of Theorem 35 suggests an interesting *alternative*.

**Corollary 36** *Let  $(\Omega, \mu, d)$  be a metric measure space. We assume that (12) or (14) is satisfied (e.g.  $\{U(t); t \geq 0\}$  is holomorphic). Let  $t > 0$  be fixed and let  $U(t) : L^1(\Omega) \rightarrow L^1(\Xi)$  be weakly compact for any bounded Borel set  $\Xi$ . We assume that the kernel  $p_t(x, y)$  of  $U(t)$  satisfies the estimate*

$$\sup_{M > 0} \lim_{C \rightarrow +\infty} \sup_{y \in \Omega} \int_{\{x \in \Omega_M; d(x, x_0) \geq C\}} p_t(x, y) \mu(dx) < e^{\lambda_1 t}$$

(for some  $x_0 \in \Omega$ ) where  $\lambda_1$  be the spectral bound of  $T$ . Then either  $\lambda_V < \lambda_1$  or  $\lambda_V = \lambda_1$  and  $\omega_{ess} < \lambda_V$ .

*Proof:* We have always  $\lambda_V \leq \lambda_1$ . Then either the type of  $\{U_V(t); t \geq 0\}$  (or equivalently the spectral bound of  $T_V$ ) is strictly less than  $\lambda_1$  or then  $\lambda_V = \lambda_1$  and we can of course replace  $\lambda_V$  by  $\lambda_1$  in (36) and appeal to Theorem 35. ■

In particular, if  $\{U(t); t \geq 0\}$  is a stochastic semigroup (i.e. mass preserving on the positive cone) then  $\int p_t(x, y) \mu(dx) = 1$  and  $\lambda_1 = 0$  so that we have:

**Corollary 37** *Let  $(\Omega, \mu, d)$  be a metric measure space and let  $\{U(t); t \geq 0\}$  be a stochastic semigroup (i.e. mass preserving on the positive cone). We*

assume that (12) or (14) is satisfied (e.g.  $\{U(t)\}$  is holomorphic). Let  $t > 0$  be fixed and let  $U(t) : L^1(\Omega) \rightarrow L^1(\Xi)$  be weakly compact for any bounded Borel set  $\Xi$ . We assume that the kernel  $p_t(x, y)$  of  $U(t)$  satisfies the estimate

$$\sup_{M>0} \lim_{C \rightarrow +\infty} \sup_{y \in \Omega} \int_{\{x \in \Omega_M; d(x, x_0) \geq C\}} p_t(x, y) \mu(dx) < 1$$

(for some  $x_0 \in \Omega$ ). Then either  $\lambda_V < 0$  or  $\omega_{ess} < \lambda_V = 0$ .

**Remark 38** If  $V$  is bounded then  $\Omega_M = \Omega$  for large  $M$ . In this case, one shows similarly that if

$$\lim_{C \rightarrow +\infty} \sup_{y \in \Omega} \int_{\{d(x, x_0) \geq C\}} p_t(x, y) \mu(dx) < e^{\lambda_1 t}$$

then either  $\lambda_V < \lambda_1$  or  $\lambda_V = \lambda_1$  and  $\omega_{ess} < \lambda_V$ . Actually, the unperturbed semigroup  $\{U(t); t \geq 0\}$  itself has a spectral gap and this property is inherited by  $\{U_V(t); t \geq 0\}$ .

**Remark 39** We note that if the semigroup  $\{U_V(t); t \geq 0\}$  is irreducible and  $\omega_{ess} < \lambda_V$  then  $\lambda_V$  is a strictly dominant (algebraically simple) eigenvalue of  $T_V$  and  $\lim_{t \rightarrow +\infty} \|e^{-\lambda_V t} U_V(t) - P\| = 0$  where  $P$  is the spectral projection associated to the leading eigenvalue  $\lambda_V$  (see e.g. [65] p. 343-344); in the case  $\lambda_V = 0$ , this implies the so-called “exponential return to equilibrium”.

We consider now the case where  $\{U(t); t \geq 0\}$  operates on all  $L^p(\Omega)$  ( $p \geq 1$ ); we denote it by  $\{U_p(t); t \geq 0\}$  in  $L^p(\Omega)$ . We denote by  $\{U_{pV}(t); t \geq 0\}$  the corresponding perturbed semigroup in  $L^p(\Omega)$  and wonder whether the latter has a spectral gap. Let  $\lambda_p$  be the spectral bound of the generator of  $\{U_p(t); t \geq 0\}$  and  $\lambda_{pV}$  be the spectral bound of the generator of  $\{U_{pV}(t); t \geq 0\}$  (note that  $\lambda_{1V}$  is the above  $\lambda_V$ ). We denote by  $\omega_{pess}$  the essential type of  $\{U_{pV}(t); t \geq 0\}$ .

**Theorem 40** Let  $(\Omega, \mu, d)$  be a metric measure space and  $\{U(t); t \geq 0\}$  a contraction semigroup in all  $L^p(\Omega)$  ( $p \geq 1$ ). We assume that (12) or (14) is satisfied (e.g.  $\{U(t)\}$  is holomorphic). Let  $t > 0$  be fixed and let  $U(t) : L^1(\Omega) \rightarrow L^1(\Xi)$  be compact for any bounded Borel set  $\Xi$ . We assume that the kernel  $p_t(x, y)$  of  $U(t)$  satisfies the estimate

$$\sup_{M>0} \lim_{C \rightarrow +\infty} \sup_{y \in \Omega} \int_{\{x \in \Omega_M; d(x, x_0) \geq C\}} p_t(x, y) \mu(dx) < e^{p\lambda_p t} \quad (38)$$

(for some  $x_0 \in \Omega$ ). Then either  $\lambda_{pV} < \lambda_p$  or  $\lambda_{pV} = \lambda_p$  and  $\omega_{pess} < \lambda_{pV}$ .

*Proof:* We note first that  $\lambda_{pV} \leq \lambda_p$ . If  $\lambda_{pV} < \lambda_p$  there is nothing to prove. Suppose now that  $\lambda_{pV} = \lambda_p$ . We resume the idea of proof of Theorem 35 and Corollary 36 but in  $L^p$  setting with  $p > 1$ . We decompose  $U_{pV}(t)$  as

$$\begin{aligned} U_{pV}(t) &= \chi_{\Omega_M^c} U_{pV}(t) + \chi_{\{x \in \Omega_M, d(x, x_0) \geq C\}} U_{pV}(t) \\ &\quad + \chi_{\{x \in \Omega_M, d(x, x_0) < C\}} U_{pV}(t) \end{aligned}$$

where  $\Omega_M^c$  is the complement of the sublevel set  $\Omega_M$ . We note the compactness of  $\chi_{\{x \in \Omega_M, d(x, x_0) < C\}} U_p(t)$  in  $L^p(\Omega)$  (by interpolation from the  $L^1$  compactness assumption) and then the domination

$$\chi_{\{x \in \Omega_M, d(x, x_0) < C\}} U_{pV}(t) \leq \chi_{\{x \in \Omega_M, d(x, x_0) < C\}} U_p(t)$$

shows that  $\chi_{\{x \in \Omega_M, d(x, x_0) < C\}} U_{pV}(t)$  is compact in  $L^p(\Omega)$  by Doods-Fremlin's theorem (see e.g. [2] Theorem 5.20, p. 286). Moreover, by (13) (a consequence of (12)), the  $L^1$ -operator norm of  $\chi_{\Omega_M^c} U_V(t)$  goes to zero as  $M \rightarrow +\infty$  and its  $L^\infty$ -operator norm is less than or equal to one. Then, by Riesz-Thorin interpolation theorem, the  $L^p$ -operator norm of  $\chi_{\Omega_M^c} U_{pV}(t)$  goes also to zero as  $M \rightarrow +\infty$ . Finally, the  $L^1$ -operator norm of  $\chi_{\{x \in \Omega_M, d(x, x_0) \geq C\}} U_V(t)$  is less than or equal to that of  $\chi_{\{x \in \Omega_M, d(x, x_0) \geq C\}} U(t)$  i.e.

$$\sup_{y \in \Omega} \int_{\{x \in \Omega_M; d(x, x_0) \geq C\}} p_t(x, y) \mu(dx)$$

(and its  $L^\infty$ -operator norm is less than or equal to one) so that, by Riesz-Thorin interpolation theorem, the  $L^p$ -operator norm of  $\chi_{\{x \in \Omega_M, d(x, x_0) \geq C\}} U_{pV}(t)$  is less than or equal to

$$\left( \sup_{y \in \Omega} \int_{\{x \in \Omega_M; d(x, x_0) \geq C\}} p_t(x, y) \mu(dx) \right)^{\frac{1}{p}}.$$

It follows that for  $M$  and  $C$  large enough the  $L^p$ -operator norm of  $\chi_{\Omega_M^c} U_{pV}(t) + \chi_{\{x \in \Omega_M, d(x, x_0) \geq C\}} U_{pV}(t)$  is less than  $(e^{p\lambda_p t})^{\frac{1}{p}} = e^{\lambda_p t}$ . Then the stability of the essential spectrum by compact perturbations shows that

$$\begin{aligned} e^{\omega_{pess} t} &= r_{ess} [U_{pV}(t)] = r_{ess} \left[ (\chi_{\Omega_M^c} + \chi_{\{x \in \Omega_M, d(x, x_0) \geq C\}}) U_{pV}(t) \right] \\ &\leq \left\| (\chi_{\Omega_M^c} + \chi_{\{x \in \Omega_M, d(x, x_0) \geq C\}}) U_{pV}(t) \right\| < e^{\lambda_p t} = e^{\lambda_{pV} t} \end{aligned}$$

so that  $\omega_{pess} < \lambda_{pV}$ . ■

**Remark 41** *If the spectral bound  $\lambda_p$  of  $T_p$  is  $p$ -independent (which occurs for some Schrödinger type semigroups [80]) then, unless  $\lambda_p = 0$ , Assumption (38) is stronger and stronger as  $p$  increases. Does this suggest that the existence of a spectral gap may depend on  $p$ ? Note that a priori the spectrum need not be  $p$ -independent (see e.g. Laplace Beltrami operator on hyperbolic space [18] p. 177); this is however the case (in metric spaces with polynomial volume growth and) under Gaussian estimates, see [19] and references therein.*

We come back to the  $L^1$ -theory. We can avoid the use of the (a priori) *unknown* parameter  $\lambda_V$  at least for symmetric sub-Markov semigroups. We restrict ourselves to Theorem 35. A basic observation is that the parameter

$$\sup_{M>0} \lim_{C \rightarrow +\infty} \sup_{y \in \Omega} \int_{\{x \in \Omega_M; d(x, x_0) \geq C\}} p_t(x, y) \mu(dx) \quad (39)$$

depends only on the values of the potential  $V$  “at infinity”, (i.e. it is independent of the value of  $V$  on the balls  $B(x_0, r)$  with finite radius). It suffices then to estimate  $\lambda_V$  by suitable parameters depending on the value of  $V$  at “finite distance” only. In this way, we will be able in principle to check the validity of (36). We start with:

**Lemma 42** *Let  $\{U_2(t); t \geq 0\}$  be symmetric. Let  $\lambda_{pV}$  be the spectral bound of  $T_{pV}$  the generator of  $\{U_{pV}(t); t \geq 0\}$  in  $L^p(\Omega; \mu)$ . Then  $\lambda_{pV} \leq \lambda_V$ .*

*Proof:* Since the spectral bound  $\lambda_V$  is equal to the type of  $\{U_V(t); t \geq 0\}$  then, for any  $\alpha > \lambda_V$  there exists  $c_\alpha \geq 1$  such that  $\|U_V(t)\|_{\mathcal{L}(L^1(\Omega; \mu))} \leq c_\alpha e^{\alpha t}$  and by symmetry  $\|U_V(t)\|_{\mathcal{L}(L^\infty(\Omega; \mu))} \leq c_\alpha e^{\alpha t}$ . Then Riesz-Thorin interpolation theorem implies  $\|U_{pV}(t)\|_{\mathcal{L}(L^p(\Omega; \mu))} \leq c_\alpha e^{\alpha t}$ . Since the spectral bound of the generator coincides with the type of the corresponding semigroup for positive semigroups in  $L^p(\Omega; \mu)$  then the type of  $\{U_{pV}(t); t \geq 0\}$  (or equivalently  $\lambda_{pV}$ ) is less than or equal to  $\alpha$  for any  $\alpha > \lambda_V$ . Hence  $\lambda_{pV} \leq \lambda_V$ . ■

Thus (36) is satisfied if

$$\sup_{M>0} \lim_{C \rightarrow +\infty} \sup_{y \in \Omega} \int_{\{x \in \Omega_M; d(x, x_0) \geq C\}} p_t(x, y) \mu(dx) < e^{\lambda_{2V} t}$$

where  $\lambda_{2V}$  is the spectral bound of  $T_{2V}$ . It suffices to derive lower bounds of  $\lambda_{2V}$  independent of the values of  $V$  “at infinity”. Note that  $-T_{2V}$  is the self-adjoint operator defined by the form

$$\left\| \sqrt{-T_2} \varphi \right\|_{L^2(\Omega; \mu)}^2 + \int V(x) |\varphi(x)|^2 \mu(dx)$$

with domain

$$D := \left\{ \varphi \in D(\sqrt{-T_2}), \int V(x) |\varphi(x)|^2 \mu(dx) < +\infty \right\}$$

so that

$$-\lambda_{2V} = \inf_{\varphi \in D, \|\varphi\|_{L^2(\Omega; \mu)}=1} \left\| \sqrt{-T_2} \varphi \right\|_{L^2(\Omega; \mu)}^2 + \int V(x) |\varphi(x)|^2 \mu(dx).$$

We define  $D_r$  as the subspace of  $D$  consisting of those elements with supports included in  $B(x_0, r)$ . We assume that

$$\exists \bar{r} > 0, D_{\bar{r}} \neq \{0\} \quad (40)$$

(in our abstract setting, this assumption seems to be necessary). Then  $D_r \neq \{0\}$  for  $r \geq \bar{r}$  and

$$-\lambda_{2V} \leq -\hat{\lambda}_r := \inf_{\varphi \in D_r, \|\varphi\|_{L^2(\Omega; \mu)}=1} \left\| \sqrt{-T_2} \varphi \right\|_{L^2(\Omega; \mu)}^2 + \int_{B(x_0, r)} V(x) |\varphi(x)|^2 \mu(dx).$$

We note that  $\hat{\lambda}_r$  does not depend on the values of  $V$  outside the ball  $B(x_0, r)$  and that  $r \rightarrow \hat{\lambda}_r$  is nondecreasing. Let  $\hat{\lambda} := \lim_{r \rightarrow +\infty} \hat{\lambda}_r$ . (We do not know a priori whether  $\hat{\lambda} = \lambda_{2V}$ .) We note that  $\hat{\lambda}$  and (39) are in some sense *independent* parameters. Hence we can state:

**Theorem 43** *Let  $\{U_2(t); t \geq 0\}$  be symmetric and (40) be satisfied. We assume that (12) or (14) is satisfied (e.g.  $\{U(t); t \geq 0\}$  is holomorphic). Let  $t > 0$  be fixed and let  $U(t) : L^1(\Omega) \rightarrow L^1(\Xi)$  be weakly compact for any bounded Borel set  $\Xi$ . If the kernel  $p_t(x, y)$  of  $U(t)$  satisfies the estimate*

$$\sup_{M>0} \lim_{C \rightarrow +\infty} \sup_{y \in \Omega} \int_{\{x \in \Omega_M; d(x, x_0) \geq C\}} p_t(x, y) \mu(dx) < e^{\hat{\lambda}t}$$

*then  $\omega_{ess} < \lambda_V$ .*

## 6 On magnetic Schrödinger operators

As noted in the Introduction, it is possible, in our general formalism, to relax the positivity assumption on the unperturbed semigroup under consideration. Indeed, since a general (i.e. non positivity preserving) contraction semigroup in  $L^1$  spaces admits a *modulus* contraction semigroup [45], we

could make the appropriate assumptions rather on the modulus of the unperturbed semigroup (or at least on some other dominating semigroup); we do not try to elaborate on this point here in full generality and restrict ourselves to the significant example of Schrödinger operators with magnetic fields

$$-(\nabla - ia)^2 + V = -\sum_{j=1}^N (\partial_j - ia_j)^2 + V \quad (41)$$

where

$$a = (a_1, a_2, \dots, a_N) \in (L_{loc}^2(\mathbb{R}^N))^N, \quad 0 \leq V \in L_{loc}^1(\mathbb{R}^N).$$

Following [50], (41) defines a self-adjoint operator  $-H_{a,V}$  on  $L^2(\mathbb{R}^N)$  by means of the closed lower bounded form

$$h(f, g) = \int_{\mathbb{R}^N} \nabla_a f \cdot \overline{\nabla_a g} dx + \int_{\mathbb{R}^N} V f \bar{g} dx$$

(where  $\nabla_a f := \nabla f - ia f$  is the generalized gradient) with domain

$$D(h) = \left\{ f \in L^2(\mathbb{R}^N); \nabla_a f \in L^2(\mathbb{R}^N), V^{\frac{1}{2}} f \in L^2(\mathbb{R}^N) \right\}$$

endowed with the norm

$$\|f\|_h = \sqrt{\|f\|^2 + \|\nabla_a f\|^2 + \|V^{\frac{1}{2}} f\|^2}.$$

It is known ([8] Theorem 2.7) that  $H_{a,V}$  is resolvent compact in  $L^2(\mathbb{R}^N)$  if  $H_{0,V}$  is; the key point being the diamagnetic pointwise estimate (42) (see [46] for more recent developments). We provide here an  $L^1$  approach which complements the known hilbertian results. We recall first the diamagnetic pointwise estimate [50]:

$$|(\lambda - H_{a,V})^{-1} f| \leq (\lambda - H_{0,V})^{-1} |f|; \quad f \in L^2(\mathbb{R}^N) \quad (\lambda > 0). \quad (42)$$

Exponential formula and (42) imply

$$|e^{tH_{a,V}} f| \leq e^{tH_{0,V}} |f|; \quad f \in L^2(\mathbb{R}^N) \quad (43)$$

we note (symbolically)  $|e^{tH_{a,V}}| \leq e^{tH_{0,V}}$ . Since  $e^{tH_{0,V}} = e^{t\Delta_V}$  leaves invariant all  $L^p(\mathbb{R}^N)$  ( $1 \leq p < +\infty$ ) then (43) shows that  $e^{tH_{a,V}}$  extends as a  $c_0$ -semigroup  $\{U_{pV}^a(t); t \geq 0\}$  in  $L^p(\mathbb{R}^N)$  ( $1 \leq p < +\infty$ ) still satisfying the estimate (43) with  $f \in L^p(\mathbb{R}^N)$ .

**Theorem 44** *We assume that the sublevel sets  $\Omega_M$  are “thin at infinity” in the sense (18). Then  $\{U_{pV}^a(t); t \geq 0\}$  is a compact semigroup in  $L^p(\mathbb{R}^N)$  ( $1 \leq p < +\infty$ ).*

*Proof:* By Corollary 13 and Theorem 21,  $e^{tH_{0,V}}$  is compact so, (43) in  $L^1$  spaces implies that  $\{U_{1V}^a(t); t \geq 0\}$  is weakly compact in  $L^1(\mathbb{R}^N)$  by domination. By Dunford-Pettis’ theorem (see e.g. [2] Corollary 5.88, p. 344)  $U_{1V}^a(t) = U_{1V}^a(\frac{t}{2})U_{1V}^a(\frac{t}{2})$  is compact in  $L^1(\mathbb{R}^N)$  as a product of two weakly compact operators. An interpolation argument shows that  $\{U_{pV}^a(t); t \geq 0\}$  is compact in all  $L^p(\mathbb{R}^N)$  ( $1 \leq p < +\infty$ ). ■

We show now the existence of a spectral gap when the sublevel sets  $\Omega_M$  are not “thin at infinity”. As far as we know, this problem has not been considered in the literature. This result covers both cases  $a \neq 0$  (the magnetic case) and  $a = 0$ .

**Theorem 45** *Let  $\lambda_V^a$  be the spectral bound of  $H_{a,V}$  and let  $\lambda_{essV}^a$  be its essential spectral bound in  $L^2(\mathbb{R}^N)$ . We assume that*

$$\sup_{M>0} \lim_{C \rightarrow +\infty} \sup_{y \in \mathbb{R}^N} \int_{\{x \in \Omega_M; |x| \geq C\}} \frac{1}{(4\pi t)^{\frac{n}{2}}} \exp\left(-\frac{|x-y|^2}{4t}\right) dx < e^{\hat{\lambda}t}, \quad (44)$$

where  $\hat{\lambda} := \lim_{r \rightarrow +\infty} \hat{\lambda}_r$  and

$$-\hat{\lambda}_r = \inf_{\varphi \in D_r(h), \|\varphi\|_{L^2}=1} \int_{B(0,r)} |\nabla_a \varphi(x)|^2 dx + \int_{B(0,r)} V(x) |\varphi(x)|^2 dx$$

where  $D_r(h)$  are the elements of  $D(h)$  with support in the ball  $B(0,r)$ . Then  $\lambda_{essV}^a < \lambda_V^a$ .

*Proof:* The essential spectral bound of  $H_{a,V}$  coincides with the essential type of  $\{e^{tH_{a,V}}; t \geq 0\}$ . We note that

$$\begin{aligned} -\lambda_V^a &= \inf_{\varphi \in D(h), \|\varphi\|_{L^2}=1} \int |\nabla_a \varphi(x)|^2 dx + \int V(x) |\varphi(x)|^2 dx \\ &\leq -\hat{\lambda}_r = \inf_{\varphi \in D_r(h), \|\varphi\|_{L^2}=1} \int_{B(0,r)} |\nabla_a \varphi(x)|^2 dx + \int_{B(0,r)} V(x) |\varphi(x)|^2 dx, \end{aligned}$$

and that  $r(\geq \bar{r}) \rightarrow \hat{\lambda}_r$  is nondecreasing so that  $\hat{\lambda} \leq \lambda_V^a$ . In particular

$$\sup_{M>0} \lim_{C \rightarrow +\infty} \sup_{y \in \mathbb{R}^N} \int_{\{x \in \Omega_M; |x| \geq C\}} \frac{1}{(4\pi t)^{\frac{n}{2}}} \exp\left(-\frac{|x-y|^2}{4t}\right) dx < e^{\lambda_V^a t}.$$

We note that the Gaussian estimate behind (43) implies that the (essential) spectrum of  $H_{aV}$  is the same in all  $L^p(\mathbb{R}^N)$  (see e.g. [19]); in particular,  $\lambda_{essV}^a$  and  $\lambda_V^a$  are  $p$ -independent. Thus, it suffices to deal with the  $L^1$  case. Let  $\frac{1}{(4\pi t)^{\frac{n}{2}}} \exp(-\frac{|x-y|^2}{4t})$  be the kernel of  $e^{t\Delta}$  (i.e. the heat kernel on  $\mathbb{R}^N$ ). Estimate (43) gives

$$|e^{tH_{a,V}}| \leq e^{tH_{0,V}} \leq e^{t\Delta} \quad (45)$$

We decompose  $e^{tH_{a,V}}$  as

$$e^{tH_{a,V}} = \chi_{\Omega_M^c} e^{tH_{a,V}} + \chi_{\{x \in \Omega_M, |x| \geq C\}} e^{tH_{a,V}} + \chi_{\{x \in \Omega_M, |x| < C\}} e^{tH_{a,V}} \quad (46)$$

where  $\Omega_M^c$  is the complement of the sublevel set  $\Omega_M$ . Since

$$\chi_{\{x \in \Omega_M, |x| < C\}} |e^{tH_{a,V}}| \leq \chi_{\{x \in \Omega_M, |x| < C\}} e^{t\Delta}$$

and  $e^{t\Delta} : L^1(\mathbb{R}^N) \rightarrow L_{loc}^1(\mathbb{R}^N)$  is weakly compact then the third operator in (46) is weakly compact. Moreover,  $\{e^{tH_{0,V}}; t \geq 0\}$  is holomorphic in  $L^1(\mathbb{R}^N)$  because the heat semigroup is and then  $\{e^{tH_{0,V}}; t \geq 0\}$  satisfies the “weak type” estimate (13) so that the operator norm of  $\chi_{\Omega_M^c} e^{tH_{0,V}}$  goes to zero as  $M \rightarrow +\infty$  and this implies that the operator norm of  $\chi_{\Omega_M^c} e^{tH_{a,V}}$  goes to zero as  $M \rightarrow +\infty$  as well. Finally, the operator norm of  $\chi_{\{x \in \Omega_M, |x| \geq C\}} e^{tH_{a,V}}$  is less than or equal to that of  $\chi_{\{x \in \Omega_M, |x| \geq C\}} e^{t\Delta}$  i.e.

$$\sup_{y \in \mathbb{R}^N} \int_{\{x \in \Omega_M; |x| \geq C\}} \frac{1}{(4\pi t)^{\frac{n}{2}}} \exp(-\frac{|x-y|^2}{4t}) dx.$$

Thus, (44) shows that for  $M$  and  $C$  large enough, the operator norm of  $\chi_{\Omega_M^c} e^{tH_{a,V}} + \chi_{\{x \in \Omega_M, |x| \geq C\}} e^{tH_{a,V}}$  is less than  $e^{\lambda_V^a t}$ . On the other hand, since  $\chi_{\{x \in \Omega_M, |x| < C\}} e^{tH_{a,V}}$  is weakly compact then the stability of the essential spectrum by weakly compact perturbations implies

$$\begin{aligned} e^{\lambda_{essV}^a t} &= r_{ess} [e^{tH_{a,V}}] = r_{ess} [\chi_{\Omega_M^c} e^{tH_{a,V}} + \chi_{\{x \in \Omega_M, |x| \geq C\}} e^{tH_{a,V}}] \\ &\leq \left\| \chi_{\Omega_M^c} e^{tH_{a,V}} + \chi_{\{x \in \Omega_M, |x| \geq C\}} e^{tH_{a,V}} \right\| < e^{\lambda_V^a t} \end{aligned}$$

and this ends the proof. ■

## 7 On weighted Laplacians

In this section, we revisit some aspects of a topic related to some of the previous results. Let  $h \in C^2(\mathbb{R}^N)$  such that  $h(x) > 0 \forall x \in \mathbb{R}^N$  and let



$\mu(dx) = h^2(x)dx$ . We define the weighted Laplacian

$$\Delta^\mu := \frac{1}{h^2} \operatorname{div}(h^2 \nabla) = \Delta + 2 \frac{\nabla h \cdot \nabla}{h}.$$

This is (minus) the self-adjoint operator in  $L^2(\mathbb{R}^N; \mu(dx))$  associated to the Dirichlet form  $\int_{\mathbb{R}^N} |\nabla \varphi|^2 \mu(dx)$ , (see e.g. [18] Section 4.7, [31]). It is easy to see that if  $V \in C(\mathbb{R}^N)$  is such that  $\Delta h = Vh$  then

$$\begin{aligned} \Delta^\mu \varphi &= \Delta \varphi + 2 \frac{\nabla h \cdot \nabla \varphi}{h} = \frac{1}{h} [h \Delta \varphi + 2 \nabla h \cdot \nabla \varphi + \varphi \Delta h - V \varphi h] \\ &= \frac{1}{h} [\Delta \varphi h - V \varphi h] \end{aligned}$$

i.e.

$$\Delta^\mu = \frac{1}{h} \circ (\Delta - V) \circ h.$$

Thus the weighted Laplacian  $\Delta^\mu$  in  $L^2(\mathbb{R}^N; \mu(dx))$  is unitarily equivalent to the Schrödinger operator  $\Delta - \frac{\Delta h}{h}$  on  $L^2(\mathbb{R}^N; dx)$  by the unitary transformation

$$I : \varphi \in L^2(\mathbb{R}^N; \mu(dx)) \rightarrow h\varphi \in L^2(\mathbb{R}^N; dx).$$

This shows that the weighted Laplacian  $\Delta^\mu$  in  $L^2(\mathbb{R}^N; \mu(dx))$  has the same spectral properties as the Schrödinger operator  $\Delta - \frac{\Delta h}{h}$  on  $L^2(\mathbb{R}^N; dx)$ ; (similar calculations can be performed by replacing the Laplacian by more general elliptic operators with smooth coefficients [17] but we restrict ourselves to this model case). We start with the following result already obtained in [54] by other means.

**Theorem 46** *Let  $h \in C^2(\mathbb{R}^N)$  with  $h(x) > 0 \ \forall x \in \mathbb{R}^N$ . We assume that  $\frac{\Delta h}{h}$  is bounded below. Then the weighted Laplacian  $\Delta^\mu$  generates a compact semigroup on  $L^2(\mathbb{R}^N; \mu(dx))$  provided that the sublevel sets  $\Omega_M$  of  $\frac{\Delta h}{h}$  are “thin at infinity”.*

*Proof:* Let  $V := \frac{\Delta h}{h}$ . If we consider the operator  $\Delta - V$  then, up to a bounded perturbation, we can assume that  $V \geq 0$ . Then, by Theorem 1,  $\Delta - V$  generates a compact semigroup on  $L^2(\mathbb{R}^N; dx)$  and we conclude by a similarity argument. ■

Generally the function  $h$  is written in the form  $h(x) := e^{-\frac{\Phi}{2}(x)}$  where  $\Phi$  be a real  $C^2$  function on  $\mathbb{R}^N$ , i.e.  $\mu(dx) = e^{-\Phi(x)}dx$ . Note that in this case

$$\Delta^\mu = \Delta + 2 \frac{\nabla h \cdot \nabla}{h} = \Delta - \nabla \Phi \cdot \nabla$$

in  $L^2(\mathbb{R}^N; e^{-\Phi(x)} dx)$ ; we do not assume a priori here the finiteness of  $\mu(dx)$ , i.e. we do not assume that  $e^{-\Phi(x)}$  is integrable. It is well known that

$$V := \frac{\Delta h}{h} = \frac{1}{4} |\nabla \Phi(x)|^2 - \frac{1}{2} \Delta \Phi(x).$$

The (minus) Schrödinger operators

$$\Delta_\Phi := -\Delta + \frac{1}{4} |\nabla \Phi|^2 - \frac{1}{2} \Delta \Phi$$

in  $L^2(\mathbb{R}^N; dx)$  are also known as the Witten Laplacians (on 0-forms) and were studied in particular in [38] in connection with Fokker-Planck operators. Thus Theorem 46 takes the form:

**Corollary 47** *Let  $\Phi$  be a real  $C^2$  function on  $\mathbb{R}^N$ . We assume that  $\frac{1}{4} |\nabla \Phi|^2 - \frac{1}{2} \Delta \Phi$  is bounded below. Then the weighted Laplacian  $\Delta^\mu$  on  $L^2(\mathbb{R}^N; \mu(dx))$  (where  $\mu(dx) = e^{-\Phi(x)} dx$ ) generates a compact semigroup provided that the sublevel sets of  $\frac{1}{4} |\nabla \Phi|^2 - \frac{1}{2} \Delta \Phi$  are “thin at infinity”.*

**Remark 48** (i) *The Ornstein-Uhlenbeck generator  $\Delta - x \cdot \nabla$  is a weighted Laplacian in  $L^2(\mathbb{R}^N; e^{-\frac{|x|^2}{2}} dx)$  unitarily equivalent to (minus)  $-\Delta + \frac{|x|^2}{4} - \frac{N}{2}$  (the harmonic oscillator) in  $L^2(\mathbb{R}^N; dx)$  and is known to generate a compact semigroup. We point out that the Ornstein-Uhlenbeck semigroup is not compact in  $L^1(\mathbb{R}^N; e^{-\frac{|x|^2}{2}} dx)$  (see [18] Section 4.3) while the semigroup generated by (minus) the harmonic oscillator is compact in  $L^1(\mathbb{R}^N; dx)$  (see Remark 26(i)).*

(ii) *We note that by construction  $-\Delta + \frac{1}{4} |\nabla \Phi|^2 - \frac{1}{2} \Delta \Phi$  is a nonnegative operator in  $L^2(\mathbb{R}^N; dx)$  even if  $\frac{1}{4} |\nabla \Phi|^2 - \frac{1}{2} \Delta \Phi$  is not bounded below. Our assumption that  $\frac{1}{4} |\nabla \Phi|^2 - \frac{1}{2} \Delta \Phi$  is bounded below is technical and is due to our  $L^1$  approach of the problem. In ([38] Proposition 3.1, p. 21) we can reach the conclusion of Corollary 47 without this assumption (by means of hypoelliptic techniques) provided there exists some  $t \in (1, 2)$  such that  $t |\nabla \Phi(x)|^2 - \Delta \Phi(x) \rightarrow +\infty$  as  $|x| \rightarrow \infty$ .*

The following (non-convex) potential appears e.g. in [36][43]

$$\Phi(x) = \frac{1}{h} \sum_{j=1}^N \left( \frac{\lambda}{12} x_j^4 + \frac{\nu}{2} x_j^2 \right) + \frac{1}{h} \frac{I}{2} \sum_{j=1}^N |x_j - x_{j+1}|^2 \quad (47)$$

(with the convention  $x_{N+1} = x_1$ ) where  $h > 0$ ,  $\lambda > 0$ ,  $\nu < 0$ ,  $I > 0$ .

**Corollary 49** *Let  $\Phi$  be of the form (47). Then  $-\Delta_\Phi$  generates a (holomorphic) compact semigroup in  $L^1(\mathbb{R}^N; dx)$ .*

*Proof:* Writing (47) in the form

$$\Phi(x) = \alpha \sum_{j=1}^N x_j^4 - \beta \sum_{j=1}^N x_j^2 + \gamma \sum_{j=1}^N |x_j - x_{j+1}|^2$$

where  $\alpha > 0$ ,  $\beta > 0$ ,  $\gamma > 0$ , it is easy to see that

$$\Delta\Phi = 12\alpha |x|^2 + \gamma(4 - 2\beta)N.$$

On the other (see [43]) there exists  $c > 0$  such that  $\nabla\Phi(x) \cdot x \geq c|x|^4$  for  $|x|$  large enough. Thus  $\nabla\Phi(x) \cdot \frac{x}{|x|} \geq c|x|^3$  and then  $|\nabla\Phi(x)| \geq c|x|^3$  for  $|x|$  large enough. Finally

$$\frac{1}{4} |\nabla\Phi|^2 - \frac{1}{2} \Delta\Phi \geq \frac{c^2 |x|^6}{4} - 6\alpha |x|^2 + \gamma(2 - \beta)N \rightarrow +\infty$$

as  $|x| \rightarrow +\infty$  and we are done. ■

Sometimes  $\Phi$  enjoys useful decompositions. We give a result in this direction and then apply it to uniformly strictly convex  $\Phi$ .

**Corollary 50** *Let  $\Phi = \Phi_1 + \Phi_2$  where  $\Phi_1, \Phi_2$  be  $C^2$  functions such that  $(\frac{|\nabla\Phi_1|^2}{4} - \frac{1}{2}\Delta\Phi_1) + \frac{1}{2}\nabla\Phi_1(x) \cdot \nabla\Phi_2(x)$  and  $\frac{|\nabla\Phi_2(x)|^2}{4} - \frac{1}{2}\Delta\Phi_2$  are bounded below. If the sublevel sets of  $\frac{|\nabla\Phi_2(x)|^2}{4} - \frac{1}{2}\Delta\Phi_2$  are “thin at infinity” then  $-\Delta_\Phi$  generates a (holomorphic) compact semigroup in  $L^1(\mathbb{R}^N; dx)$ .*

*Proof:* We note that

$$\Delta_\Phi := -\Delta + \left(\frac{|\nabla\Phi_1|^2}{4} - \frac{1}{2}\Delta\Phi_1\right) + \left(\frac{|\nabla\Phi_2(x)|^2}{4} - \frac{1}{2}\Delta\Phi_2\right) + \frac{1}{2}\nabla\Phi_1(x) \cdot \nabla\Phi_2(x).$$

We may assume that  $(\frac{|\nabla\Phi_1|^2}{4} - \frac{1}{2}\Delta\Phi_1) + \frac{1}{2}\nabla\Phi_1(x) \cdot \nabla\Phi_2(x)$  and  $\frac{|\nabla\Phi_2(x)|^2}{4} - \frac{1}{2}\Delta\Phi_2$  are nonnegative. One sees that the sublevel sets of  $\frac{|\nabla\Phi(x)|^2}{4} - \frac{1}{2}\Delta\Phi$  are included in the sublevel sets  $\frac{|\nabla\Phi_2(x)|^2}{4} - \frac{1}{2}\Delta\Phi_2$  and then are “thin at infinity” whence  $-\Delta_\Phi$  generates a (holomorphic) compact semigroup in  $L^1(\mathbb{R}^N; dx)$ . ■

A classical result by D. Bakry and M. Emery (see e.g. [72] Théorème 3.1.29, p. 50) asserts that if  $\Phi$  is uniformly strictly convex with  $\int e^{-\Phi(x)} dx = 1$  then the probability measure  $\mu(dx) = e^{-\Phi(x)} dx$  satisfies a logarithmic-Sobolev (or Gross) inequality and consequently (see e.g. [72] Proposition 3.1.8, p. 37) the spectral gap (or Poincaré) inequality holds. We complement this by the following result which does not rely on the integrability of  $e^{-\Phi(x)}$ :

**Corollary 51** *Let  $\Phi$  be uniformly strictly convex such that  $\frac{1}{4}|\nabla\Phi|^2 - \frac{1}{2}\Delta\Phi$  is bounded below. Then  $-\Delta_\Phi$  generates a (holomorphic) compact semigroup in  $L^1(\mathbb{R}^N; dx)$ .*

*Proof:* By assumption, there exists  $m > 0$  such that  $\Phi''(x) \geq mI$  ( $\Phi''(x)$  is the Hessian of  $\Phi$  at  $x$ ). Let  $\Phi_1(x) = \Phi(x) - \frac{m}{3}|x|^2$ . Then  $\Phi_1''(x)(h, h) = \Phi''(x)(h, h) - \frac{2m}{3}|h|^2 \geq \frac{m}{3}|h|^2$ , i.e.  $\Phi_1''(x) \geq \frac{m}{3}I$  so  $\Phi_1$  is uniformly strictly convex and consequently (see e.g. [72] p. 48)  $x \cdot \nabla\Phi_1(x) \geq \frac{m}{3}|x|^2 - b$  where  $b$  is a constant. Thus  $\Phi(x) = \Phi_1(x) + \Phi_2(x)$  (where  $\Phi_2(x) = \frac{m}{3}|x|^2$ ) with  $\nabla\Phi_1(x) \cdot \nabla\Phi_2(x) = \frac{2m}{3}x \cdot \nabla\Phi_1(x) \geq \frac{2m^2}{9}|x|^2 - \frac{2m}{3}b$ . It follows that  $\frac{|\nabla\Phi_1|^2}{4} - \frac{1}{2}\Delta\Phi_1$  is bounded from below since  $\frac{|\nabla\Phi_2(x)|^2}{4} - \frac{1}{2}\Delta\Phi_2 = \frac{m^2}{9}|x|^2 - \frac{mN}{2}$  is. This ends the proof since  $\frac{|\nabla\Phi_2(x)|^2}{4} - \frac{1}{2}\Delta\Phi_2 \rightarrow +\infty$  as  $|x| \rightarrow \infty$ . ■

We find in [38] systematic results on resolvent compactness or spectral gaps when  $\Phi$  is a polynomial. In particular, if  $\Phi$  is a sum of nonpositive monomials then  $\Delta_\Phi$  is resolvent compact in  $L^2(\mathbb{R}^N; dx)$  if and only if  $\sum_{|\alpha|>0} |D_x^\alpha \Phi(x)| \rightarrow +\infty$  as  $|x| \rightarrow +\infty$ , see [38] Theorem 11.10 (ii), p. 120. We complement this by:

**Theorem 52** *Let*

$$\Phi(x) = - \sum_{|\alpha| \leq C} c_\alpha x_1^{2\alpha_1} x_2^{2\alpha_2} \dots x_N^{2\alpha_N}, \quad (c_\alpha > 0) \quad (48)$$

*where  $\bar{\alpha}_i > 0 \ \forall i$  for at least one multi-index  $\bar{\alpha}$ . Then  $-\Delta_\Phi$  generates a (holomorphic) compact semigroup in  $L^1(\mathbb{R}^N)$ .*

*Proof:* We have

$$\begin{aligned} \frac{\partial\Phi}{\partial x_j} &= - \sum_{|\alpha| \leq C} (2\alpha_j c_\alpha) x_j^{2\alpha_j-1} \prod_{i \neq j} x_i^{2\alpha_i} \\ \frac{\partial^2\Phi}{\partial x_j^2} &= - \sum_{|\alpha| \leq C} (2\alpha_j - 1)(2\alpha_j c_\alpha) x_j^{2\alpha_j-2} \prod_{i \neq j} x_i^{2\alpha_i} \leq 0 \end{aligned}$$

so that  $-\Delta\Phi \geq 0$ . On the other hand

$$\begin{aligned}
|\nabla\Phi|^2 &= \sum_{j=1}^N \left[ \sum_{|\alpha| \leq C} (2\alpha_j c_\alpha) x_j^{2\alpha_j-1} \prod_{i \neq j} x_i^{2\alpha_i} \right]^2 \\
&\geq \sum_{j=1}^N \sum_{|\alpha| \leq C} (2\alpha_j c_\alpha)^2 x_j^{2(2\alpha_j-1)} \prod_{i \neq j} x_i^{4\alpha_i} \\
&\geq \sum_{j=1}^N (2\bar{\alpha}_j c_{\bar{\alpha}})^2 x_j^{2(2\bar{\alpha}_j-1)} \prod_{i \neq j} x_i^{4\bar{\alpha}_i}.
\end{aligned}$$

We observe that  $\frac{1}{4}|\nabla\Phi|^2 - \frac{1}{2}\Delta\Phi \geq 0$  and  $\left\{x; \frac{1}{4}|\nabla\Phi(x)|^2 - \frac{1}{2}\Delta\Phi(x) \leq M\right\}$  is included in

$$\left\{x; x_j^{2(2\bar{\alpha}_j-1)} \prod_{i \neq j} x_i^{4\bar{\alpha}_i} \leq \frac{4M}{(2\bar{\alpha}_j c_{\bar{\alpha}})^2}\right\}$$

for any  $j$ . It suffices to show that the latter set is thin at infinity. We may also restrict ourselves to positive coordinates. This set is defined by

$$x_j \leq \frac{M_j}{\prod_{i \neq j} x_i^{\frac{2\bar{\alpha}_i}{(2\bar{\alpha}_j-1)}}}$$

where

$$M_j = \left[ \frac{4M}{(2\bar{\alpha}_j c_{\bar{\alpha}})^2} \right]^{\frac{1}{2(2\bar{\alpha}_j-1)}}.$$

To fix the notations, suppose that  $j = N$  and set

$$\beta_i := \frac{2\bar{\alpha}_i}{(2\bar{\alpha}_N - 1)}, \quad 1 \leq i \leq N-1.$$

Note first that if  $a_N$  is large enough then the interesection of a cube

$$C := \{x; a_i - 1 \leq x_i \leq a_i + 1; \forall i\}$$

with the set defined by  $x_N \leq \frac{M_N}{\prod_{i=1}^{N-1} x_i^{\beta_i}}$  is empty. On the other hand, it is true that the Lebesgue measure of this interesection is always less than

$$\begin{aligned}
&M_N \int_{a_1-1}^{a_1+1} \frac{dx_1}{x_1^{\beta_1}} \cdots \int_{a_{N-1}-1}^{a_{N-1}+1} \frac{dx_{N-1}}{x_{N-1}^{\beta_{N-1}}} \\
&= M_N \left[ \frac{1}{(1-\beta_1)} \left( \frac{1}{(a_1-1)^{\beta_1}} - \frac{1}{(a_1+1)^{\beta_1}} \right) \right] \cdots \\
&\quad \cdots \left[ \frac{1}{(1-\beta_{N-1})} \left( \frac{1}{(a_{N-1}-1)^{\beta_1}} - \frac{1}{(a_{N-1}+1)^{\beta_1}} \right) \right]
\end{aligned}$$

when  $\beta_i \neq 1$ , otherwise replace the corresponding term by  $\ln(\frac{(a_i+1)}{(a_i+1)})$ . One sees that

$$M_N \int_{a_1-1}^{a_1+1} \frac{dx_1}{x_1^{\beta_1}} \cdots \int_{a_{N-1}-1}^{a_{N-1}+1} \frac{dx_{N-1}}{x_{N-1}^{\beta_{N-1}}} \rightarrow 0$$

if (at least) one coordinate  $a_i$  ( $1 \leq i \leq N-1$ ) tends to infinity. ■

The case of nonnegative polynomials

$$\Phi(x) = \sum_{|\alpha| \leq C} c_\alpha x_1^{2\alpha_1} x_2^{2\alpha_2} \dots x_N^{2\alpha_N}, \quad (c_\alpha > 0)$$

is much more involved even for homogeneous polynomials, see [38]. We restrict ourselves to the simplest “elliptic” case.

**Theorem 53** *Let  $\Phi(x) = \sum_{|\alpha|=r} c_\alpha x_1^{2\alpha_1} x_2^{2\alpha_2} \dots x_N^{2\alpha_N}$  ( $c_\alpha > 0$ ). If  $\nabla\Phi(x) \neq 0$  for  $x \neq 0$  then  $-\Delta_\Phi$  generates a (holomorphic) compact semigroup in  $L^1(\mathbb{R}^N)$ .*

*Proof:* It is known (see [38]) that  $\frac{1}{4}|\nabla\Phi|^2 - \frac{1}{2}\Delta\Phi \rightarrow +\infty$  as  $|x| \rightarrow \infty$ ; this is a consequence of the following facts: The compactness of the unit sphere  $S^{N-1}$  implies the existence of a constant  $c > 0$  such that  $|\nabla\Phi(x)| \geq c \forall x \in S^{N-1}$  and then  $|\nabla\Phi(x)| \geq c|x|^{2r-1} \forall x \in \mathbb{R}^N$  since  $\Phi$  is homogeneous of degree  $2r$ ; on the other hand,  $\Delta\Phi = \sum_{|\alpha|=r} \sum_{j=1}^N (2\alpha_j - 1)(2\alpha_j c_\alpha) x_j^{-2} x_1^{2\alpha_1} x_2^{2\alpha_2} \dots x_N^{2\alpha_N}$ . This ends the proof. ■

**Remark 54** (i) *Theorem 53 covers e.g. the case  $\Phi(x) = \sum_{i=1}^N c_i x_i^{2k}$  ( $c_i > 0$ ) where  $k \geq 1$ .*

(ii) *When the set of non-zero critical points of  $\Phi(x)$  is not empty (and under the non-degeneracy condition  $\sum_{1 \leq |\alpha| \leq 2} |\partial^\alpha \Phi(x)| \neq 0$  for  $x \neq 0$ )  $\Delta_\Phi$  is resolvent compact in  $L^2(\mathbb{R}^N)$  provided that for the critical points  $\omega$ , the restriction of  $\Phi''(x)$  to  $(\mathbb{R}\omega)^\perp$  is nondegenerate and not of index 0 (see [38] Proposition 10.17, p. 108). We conjecture that in this case  $-\Delta_\Phi$  generates a compact semigroup in  $L^1(\mathbb{R}^N)$  provided that  $\frac{1}{4}|\nabla\Phi|^2 - \frac{1}{2}\Delta\Phi$  is lower bounded.*

(iii) *Let  $N = 2$  and  $\Phi(x_1, x_2) = x_1^2 x_2^2 + \varepsilon(x_1^2 + x_2^2)$  ( $\varepsilon > 0$ ). It is known (see [38] Proposition 10.20, p. 111) that  $\Delta_\Phi$  is resolvent compact in  $L^2(\mathbb{R}^2)$  for all  $\varepsilon > 0$ . We can obtain a stronger conclusion for  $\varepsilon \geq 1$ . Indeed, one checks that*

$$\frac{1}{4}|\nabla\Phi|^2 - \frac{1}{2}\Delta\Phi = x_1^2 x_2^2 (x_1^2 + x_2^2 + 2\varepsilon) + (\varepsilon^2 - 1)(x_1^2 + x_2^2) - 2\varepsilon \geq x_1^2 x_2^2 (x_1^2 + x_2^2) - 2\varepsilon$$

*so that, for  $(x_1^2 + x_2^2) \geq 1$ ,  $\frac{1}{4}|\nabla\Phi|^2 - \frac{1}{2}\Delta\Phi \geq x_1^2 x_2^2 - 2\varepsilon$  and then, by Remark 26 (i),  $\Delta_\Phi$  generates a compact (holomorphic) semigroup in  $L^1(\mathbb{R}^2)$ . Note that  $\frac{1}{4}|\nabla\Phi|^2 - \frac{1}{2}\Delta\Phi$  is not bounded below if  $\varepsilon < 1$ .*

We give now an approach of spectral gaps for weighted Laplacians in terms of kernel estimates involving sublevel sets of  $\frac{1}{4}|\nabla\Phi|^2 - \frac{1}{2}\Delta\Phi$ . Two (different) results on the existence of spectral gaps are given. By adding a positive constant  $\beta$  to  $\frac{1}{4}|\nabla\Phi|^2 - \frac{1}{2}\Delta\Phi$  and shifting the Laplacien by  $-\beta$  if necessary (the heat kernel is then multiplied by  $e^{-\beta t}$ ), we may assume without loss of generality that  $\frac{1}{4}|\nabla\Phi|^2 - \frac{1}{2}\Delta\Phi$  is nonnegative.

**Theorem 55** *Let  $\Phi$  be a real  $C^2$  function on  $\mathbb{R}^N$  with  $\frac{1}{4}|\nabla\Phi|^2 - \frac{1}{2}\Delta\Phi \geq 0$  and let  $\Omega_M$  be the sublevel sets of  $\frac{1}{4}|\nabla\Phi|^2 - \frac{1}{2}\Delta\Phi$ . Then the semigroup generated by the weighted Laplacian on  $L^2(\mathbb{R}^N; \mu(dx))$  has a spectral gap provided that (for some  $t > 0$ )*

$$\sup_{M>0} \lim_{C \rightarrow +\infty} \sup_{y \in \mathbb{R}^N} \int_{\{x \in \Omega_M; |x| \geq C\}} \frac{1}{(4\pi t)^{\frac{n}{2}}} \exp\left(-\frac{|x-y|^2}{4t}\right) dx < e^{\hat{\lambda}t}, \quad (49)$$

where  $\hat{\lambda} := \lim_{r \rightarrow +\infty} \hat{\lambda}_r$  and  $-\hat{\lambda}_r$  is equal to

$$\inf_{\varphi \in H_0^1(B(0,r)), \|\varphi\|_{L^2}=1} \int_{B(0,r)} |\nabla\varphi(x)|^2 dx + \int_{B(0,r)} \left[ \frac{1}{4}|\nabla\Phi|^2 - \frac{1}{2}\Delta\Phi \right] |\varphi(x)|^2 dx.$$

*Proof:* By unitary equivalence, we have just to deal with the Schrödinger operator  $\Delta - (\frac{1}{4}|\nabla\Phi|^2 - \frac{1}{2}\Delta\Phi)$  in  $L^2(\mathbb{R}^N; dx)$ . The corresponding semigroup is dominated by the heat semigroup so that its kernel admits a Gaussian estimate and then (see e.g. [19]) its spectrum is the same in all  $L^p$  spaces. It suffices then to work in the space  $L^1(\mathbb{R}^N; dx)$  and use Theorem 43. ■

In the more usual case where  $e^{-\Phi(x)} \in L^1(\mathbb{R}^N; dx)$  we obtain a much more precise and explicit result relying on Theorem 35.

**Theorem 56** *Let  $\Phi$  be a real  $C^2$  function on  $\mathbb{R}^N$  with  $\frac{1}{4}|\nabla\Phi|^2 - \frac{1}{2}\Delta\Phi \geq 0$  and let  $\Omega_M$  be the sublevel sets of  $\frac{1}{4}|\nabla\Phi|^2 - \frac{1}{2}\Delta\Phi$ . We assume that  $e^{-\Phi(x)}$  is integrable. Then the semigroup generated by the weighted Laplacian on  $L^2(\mathbb{R}^N; \mu(dx))$  has a spectral gap provided that (for some  $t > 0$ )*

$$\sup_{M>0} \lim_{C \rightarrow +\infty} \sup_{y \in \mathbb{R}^N} \int_{\{x \in \Omega_M; |x| \geq C\}} \frac{1}{(4\pi t)^{\frac{n}{2}}} \exp\left(-\frac{|x-y|^2}{4t}\right) dx < 1. \quad (50)$$

*Proof:* When  $e^{-\Phi(x)} \in L^1(\mathbb{R}^N; dx)$  then  $\mu(dx)$  is finite and then the constant function 1 is an eigenfunction of  $\Delta^\mu$  associated to the eigenvalue 0 which is then the spectral bound of  $\Delta^\mu$ . Then 0 is also the spectral bound of  $\Delta - (\frac{1}{4}|\nabla\Phi|^2 - \frac{1}{2}\Delta\Phi)$  in  $L^2(\mathbb{R}^N; dx)$  and also in  $L^1(\mathbb{R}^N; dx)$  because the spectrum is the same in  $L^2(\mathbb{R}^N; dx)$  and  $L^1(\mathbb{R}^N; dx)$  (see e.g. [19]) whence  $\lambda_V = 0$  and we conclude by Theorem 35. ■

**Remark 57** One sees that (50) provides us with a sufficient condition in terms of sublevel sets of  $\frac{1}{4}|\nabla\Phi|^2 - \frac{1}{2}\Delta\Phi$  for the probability measure  $\mu(dx) = Z^{-1}e^{-\Phi(x)}dx$  (where  $Z = \int e^{-\Phi(x)}$ ) to satisfy the Poincaré inequality.

## 8 On Witten Laplacians on 1-forms

The De Rham Complex is given by

$$\begin{array}{ccccccc} & d^{(0)} & & d^{(1)} & & & \\ \Omega^0 & \rightarrow & \Omega^1 & \rightarrow & \Omega^2 & \rightarrow & \dots & \Omega^N & \rightarrow & 0 \end{array}$$

where  $\Omega^p := \Omega^p(\mathbb{R}^N)$  ( $p \leq N$ ) denotes the space of  $\mathcal{C}^\infty$   $p$ -forms and  $d^{(p)} : \Omega^p \rightarrow \Omega^{p+1}$  is the restriction to  $\Omega^p$  of the exterior differential  $d$ . We equip (the coefficients of the form)  $\Omega^p$  with an  $L^2$  structure and obtain the space (still denoted by  $\Omega^p$ ) of  $L^2$   $p$ -forms. We extend then  $d^{(p)}$  to  $L^2$   $p$ -forms as a closed densely defined unbounded operator  $d^{(p)} : \Omega^p \rightarrow \Omega^{p+1}$  and denote by  $d^{*(p)} : \Omega^{p+1} \rightarrow \Omega^p$  its adjoint operator. The Laplacian  $\Delta^{(p)}$  on  $\Omega^p$  is defined as

$$\Delta^{(p)} = d^{*(p)} \circ d^{(p)} + d^{(p-1)} \circ d^{*(p-1)} \quad (p \geq 1)$$

and  $\Delta^{(0)} = d^{*(0)} \circ d^{(0)}$ . In particular  $\Delta^{(0)} = -\Delta$  and  $\Delta^{(1)}\omega = \sum_{j=1}^N (\Delta^{(0)}\omega_j)dx_j$  for  $\omega = \sum_{j=1}^N \omega_j dx_j$ ; (see [79][43] and [37] Chapter 2 for the details). By replacing the exterior differential  $d$  by  $d_\Phi := e^{-\frac{\Phi}{2}(x)} d e^{\frac{\Phi}{2}(x)}$  i.e.  $d_\Phi = d + \frac{1}{2}d\Phi \wedge$ , we obtain a new Complex, the Witten Complex. By keeping the above  $L^2$  structure we can define closed unbounded operators  $d_\Phi^{(p)} : \Omega^p \rightarrow \Omega^{p+1}$  and Witten Laplacians on  $\Omega^p$

$$\Delta_\Phi^{(p)} = d_\Phi^{*(p)} \circ d_\Phi^{(p)} + d_\Phi^{(p-1)} \circ d_\Phi^{*(p-1)} \quad (p \geq 1)$$

with  $\Delta_\Phi^{(0)} = d_\Phi^{*(0)} \circ d_\Phi^{(0)}$ . In particular

$$\Delta_\Phi^{(0)} = \Delta^{(0)} + \frac{1}{4}|\nabla\Phi|^2 - \frac{1}{2}\Delta\Phi$$

and

$$\Delta_\Phi^{(1)} = \Delta_\Phi^{(0)} \otimes Id + Hess\Phi.$$

We note that  $\Delta_\Phi^{(0)}$  and  $\Delta_\Phi^{(1)}$  are lower bounded (nonnegative) operators and  $\mathcal{C}_c^\infty$  (the  $\mathcal{C}^\infty$  functions with compact support) is a core for both; see [76]. The Witten Laplacian  $\Delta_\Phi^{(0)}$  on 0-forms has been considered in the previous section. The aim of this section is to show that there exist interesting



compactness connections between  $\Delta_{\Phi}^{(0)}$  and  $\Delta_{\Phi}^{(1)}$  (see e.g. [43] Theorem 1.3 for other kinds of connections). To this end, we recall first a basic functional analytic result related to Glazman's Lemma.

**Theorem 58** ([67] Proposition 6.1.4, Corollaries 6.1.1 and 6.1.2, p. 72). *Let  $A$  and  $B$  be two self-adjoint operators in a Hilbert space  $\mathcal{H}$  such that*

$$(Au, u) \leq (Bu, u), \quad u \in \mathcal{D}$$

*where  $\mathcal{D} \subset \mathcal{H}$  is a core for both  $A$  and  $B$ . Then:*

(i) *For any real  $\lambda$ , if  $\sigma(A) \cap (-\infty, \lambda)$  is discrete (i.e. consists of isolated eigenvalues of  $A$ ) then  $\sigma(B) \cap (-\infty, \lambda)$  is also discrete (i.e. consists of isolated eigenvalues of  $B$ ).*

(ii) *If we denote by  $\lambda_1^A \leq \lambda_2^A \leq \dots \leq \lambda_k^A \leq \dots$  and  $\lambda_1^B \leq \lambda_2^B \leq \dots \leq \lambda_k^B \leq \dots$  their eigenvalues in  $(-\infty, \lambda)$ , numbered according to their multiplicities, then  $\lambda_k^A \leq \lambda_k^B$ .*

For any bounded below self-adjoint operator  $A$ , we define its essential lower bound as  $\lambda_{ess} = \sup \{ \lambda \in \mathbb{R}; \sigma(A) \cap (-\infty, \lambda) \text{ is discrete} \}$ . We start with two theorems based on a *convexity* assumption.

**Theorem 59** *We assume that  $\Phi$  is a convex  $\mathcal{C}^2$  function. Let  $\lambda_{ess}^0$  and  $\lambda_{ess}^1$  be respectively the essential lower bound of  $\Delta_{\Phi}^{(0)}$  and  $\Delta_{\Phi}^{(1)}$ . Then  $\lambda_{ess}^0 \leq \lambda_{ess}^1$ . In particular, if  $\Delta_{\Phi}^{(0)}$  is resolvent compact then  $\Delta_{\Phi}^{(1)}$  is also resolvent compact.*

*Proof:* Let  $A = \Delta_{\Phi}^{(0)} \otimes Id$  and  $B = \Delta_{\Phi}^{(1)}$ . The convexity of  $\Phi$  implies that  $Hess\Phi$  is a form-nonnegative multiplication (matrix) operator so that  $(A\omega, \omega) \leq (B\omega, \omega)$  for  $\mathcal{C}_c^\infty$  1-forms  $\omega$ . Note that  $A$  is nothing but  $N$  copies of  $\Delta_{\Phi}^{(0)}$  so that  $A$  has the same spectral structure as  $\Delta_{\Phi}^{(0)}$ . In particular, the essential lower bound of  $\Delta_{\Phi}^{(0)}$  coincides with that of  $A$ . Thus  $\sigma(A) \cap (-\infty, \lambda_{ess}^0)$  is discrete and then, by Theorem 58,  $\sigma(B) \cap (-\infty, \lambda_{ess}^0)$  is also discrete so that  $\lambda_{ess}^0 \leq \lambda_{ess}^1$ . In particular, if  $\Delta_{\Phi}^{(0)}$  is compact then  $\lambda_{ess}^0 = +\infty$  and then  $\sigma(B)$  is purely discrete, i.e.  $B$  is resolvent compact; (this last property holds without convexity assumption, see Theorem 63 below). ■

**Remark 60** *Note that below the value  $\lambda_{ess}^0$ , the  $k$ -th eigenvalue of  $\Delta_{\Phi}^{(0)}$  is majorized by that of  $\Delta_{\Phi}^{(1)}$ .*

We provide now a strict inequality between the spectral bottoms of  $\Delta_{\Phi}^{(0)}$  and  $\Delta_{\Phi}^{(1)}$ .

**Theorem 61** *Let  $\Phi$  be a convex  $\mathcal{C}^2$  function and let  $\lambda^0$  and  $\lambda^1$  be respectively the spectral bottom of  $\Delta_{\Phi}^{(0)}$  and  $\Delta_{\Phi}^{(1)}$ . We assume that  $\lambda^0$  is an isolated eigenvalue of  $\Delta_{\Phi}^{(0)}$ . If the lowest eigenvalue  $\lambda_{\Phi}(x)$  of  $\text{Hess}\Phi(x)$  is not identically zero then  $\lambda^1 > \lambda^0$ .*

*Proof:* We note that for a 1-form  $\omega(x) = \sum_{j=1}^N \omega_j(x) dx_j$  we have

$$(\text{Hess}\Phi(x)\omega(x), \omega(x))_{\mathbb{R}^N} \geq \lambda_{\Phi}(x) |\omega(x)|^2 = \lambda_{\Phi}(x) \sum_{j=1}^N \omega_j(x)^2$$

i.e.  $\text{Hess}\Phi \geq \lambda_{\Phi}(x) Id$ . It follows that

$$(\Delta_{\Phi}^{(0)} + \lambda_{\Phi}) \otimes Id \leq \Delta_{\Phi}^{(1)} \quad (51)$$

and then the spectral bottom of  $(\Delta_{\Phi}^{(0)} + \lambda_{\Phi}) \otimes Id$  (or equivalently the spectral bottom of  $\Delta_{\Phi}^{(0)} + \lambda_{\Phi}$ ) is less than or equal to the spectral bottom of  $\Delta_{\Phi}^{(1)}$ . It suffices then to compare  $\lambda^0$  the the spectral bottom  $\tilde{\lambda}^0$  of  $\Delta_{\Phi}^{(0)} + \lambda_{\Phi}$ . By the convexity of  $\Phi$  we have  $\lambda_{\Phi} \geq 0$  and then  $\Delta_{\Phi}^{(0)} \leq \Delta_{\Phi}^{(0)} + \lambda_{\Phi}$  implying the trivial inequality  $\lambda^0 \leq \tilde{\lambda}^0$ . Suppose now that  $\lambda^0$  is an isolated eigenvalue of  $\Delta_{\Phi}^{(0)}$ ; then there exists  $\alpha > 0$  such that  $\sigma(\Delta_{\Phi}^{(0)}) \cap [\lambda^0, \lambda^0 + \alpha)$  is discrete and then, by Theorem 58,  $\sigma(\Delta_{\Phi}^{(0)} + \lambda_{\Phi}) \cap [\lambda^0, \lambda^0 + \alpha)$  is also discrete (possibly empty). Thus, if  $\tilde{\lambda}^0 \geq \lambda^0 + \alpha$  we are done. Otherwise,  $\tilde{\lambda}^0$  is an isolated eigenvalue; by a classical result it is simple and is associated to a normalized positive (almost everywhere) eigenfunction  $\tilde{f}$ . By assumption, there exists also a normalized positive (almost everywhere) eigenfunction  $f$  associated to the eigenvalue  $\lambda^0$  of  $\Delta_{\Phi}^{(0)}$ . The fact that  $(f, \lambda_{\Phi} \tilde{f}) > 0$  when  $\lambda_{\Phi}(\cdot)$  is not identically zero implies

$$\lambda^0(f, \tilde{f}) = (\Delta_{\Phi}^{(0)} f, \tilde{f}) = (f, \Delta_{\Phi}^{(0)} \tilde{f}) < (f, \Delta_{\Phi}^{(0)} \tilde{f} + \lambda_{\Phi} \tilde{f}) = \tilde{\lambda}^0(f, \tilde{f})$$

so that  $\lambda^0 < \tilde{\lambda}^0$ . ■

**Remark 62** *In the case where  $\int e^{-\Phi(x)} dx = 1$  then  $\lambda^0 = 0$  and, by Theorem 61,  $\lambda^1 > 0$  so that  $\Delta_{\Phi}^{(1)}$  is invertible allowing thus the derivation of the “exact” Helffer- Sjöstrand’s covariance formula while Brascamp-Lieb’s inequality*

$$\int (f(x) - \langle f \rangle)(g(x) - \langle g \rangle) e^{-\Phi(x)} dx \leq ((\text{Hess}\Phi)^{-1} df, dg)$$

*is meaningful for strictly convex  $\Phi$  only; see [43] for more information.*

We give a compactness result for  $\Delta_{\Phi}^{(1)}$  which does not rely on a convexity assumption.

**Theorem 63** *We assume that  $\Phi$  is a  $\mathcal{C}^2$  function. Let  $\lambda_{\Phi}(x)$  be the lowest eigenvalue of  $\text{Hess}\Phi(x)$ . We assume that  $\frac{1}{4}|\nabla\Phi|^2 - \frac{1}{2}\Delta\Phi + \lambda_{\Phi}(x)$  is bounded below. Then  $\Delta_{\Phi}^{(1)}$  is resolvent compact if  $-\Delta + \frac{1}{4}|\nabla\Phi|^2 - \frac{1}{2}\Delta\Phi + \lambda_{\Phi}(x)$  is. In particular, if both  $\frac{1}{4}|\nabla\Phi|^2 - \frac{1}{2}\Delta\Phi$  and  $\lambda_{\Phi}(x)$  are bounded below then  $\Delta_{\Phi}^{(1)}$  is resolvent compact provided that  $\Delta_{\Phi}^{(0)}$  is resolvent compact or the sublevel sets of  $\lambda_{\Phi}(x)$  are thin at infinity.*

*Proof:* It follows from (51) and Theorem 58, that  $\Delta_{\Phi}^{(1)}$  is resolvent compact if  $\Delta_{\Phi}^{(0)} + \lambda_{\Phi}(x)$  is; the remainder is clear. ■

We deal now with spectral gaps for Witten Laplacians on 1-forms. We assume that  $\Phi$  is a  $\mathcal{C}^2$  function such that  $\frac{1}{4}|\nabla\Phi|^2 - \frac{1}{2}\Delta\Phi + \lambda_{\Phi}$  is bounded below; for simplicity, we assume that  $\frac{1}{4}|\nabla\Phi|^2 - \frac{1}{2}\Delta\Phi + \lambda_{\Phi} \geq 0$  (otherwise we “shift” the operator by adding a suitable constant). Let  $D^1$  be the space of 1-form  $\omega = \sum_{j=1}^N \omega_j dx_j$  with  $\omega_j \in H^1(\mathbb{R}^N)$  and

$$\sum_{j=1}^N \int \left( \frac{1}{4} |\nabla\Phi|^2 - \frac{1}{2} \Delta\Phi \right) |\omega_j(x)|^2 dx + \int (\text{Hess}\Phi(x)\omega(x), \omega(x))_{\mathbb{R}^N} dx < \infty.$$

We note that the bottom of the spectrum of  $\Delta_{\Phi}^{(1)}$  is

$$\begin{aligned} -\lambda^1 \quad : \quad &= \inf_{\omega \in D^1, \|\omega\|_{L^2}=1} \sum_{j=1}^N \left[ \int |\nabla\omega_j(x)|^2 dx + \int \left( \frac{1}{4} |\nabla\Phi|^2 - \frac{1}{2} \Delta\Phi \right) |\omega_j(x)|^2 dx \right] \\ &+ \int (\text{Hess}\Phi(x)\omega(x), \omega(x))_{\mathbb{R}^N} dx \end{aligned}$$

while the bottom of the spectrum of  $\Delta_{\Phi}^{(0)} + \lambda_{\Phi}(x)$  is

$$-\lambda^0 := \inf_{f \in D^0, \|f\|_{L^2}=1} \left[ \int |\nabla f(x)|^2 dx + \int \left( \frac{1}{4} |\nabla\Phi|^2 - \frac{1}{2} \Delta\Phi + \lambda_{\Phi} \right) |f(x)|^2 dx \right]$$

where  $D^0 = \left\{ f \in H^1(\mathbb{R}^N); \int \left( \frac{1}{4} |\nabla\Phi|^2 - \frac{1}{2} \Delta\Phi + \lambda_{\Phi} \right) |f(x)|^2 dx < \infty \right\}$ . Clearly  $-\lambda^0 \leq -\lambda^1$ . Let  $-\lambda_r^1$  be equal to

$$\begin{aligned} &\inf_{\omega \in D_r^1, \|\omega\|_{L^2}=1} \sum_{j=1}^N \left[ \int_{B(0,r)} |\nabla\omega_j(x)|^2 dx + \int_{B(0,r)} \left( \frac{1}{4} |\nabla\Phi|^2 - \frac{1}{2} \Delta\Phi \right) |\omega_j(x)|^2 dx \right] \\ &+ \int_{B(0,r)} (\text{Hess}\Phi(x)\omega(x), \omega(x))_{\mathbb{R}^N} dx \end{aligned}$$

where  $D_r^1$  are the elements of  $D^1$  with support in  $B(0, r)$ . We note that  $\lambda_r^1 \leq \lambda^1 \forall r > 0$  and  $r \rightarrow \lambda_r^1$  is nondecreasing. Let  $\hat{\lambda}^1 := \lim_{r \rightarrow +\infty} \lambda_r^1$ .

**Theorem 64** *Let  $\Phi$  be a  $\mathcal{C}^2$  function such that  $\frac{1}{4}|\nabla\Phi|^2 - \frac{1}{2}\Delta\Phi + \lambda_\Phi \geq 0$ . We denote by  $\Omega_M$  the sublevel sets of  $\frac{1}{4}|\nabla\Phi|^2 - \frac{1}{2}\Delta\Phi + \lambda_\Phi$ . We assume that for some  $t > 0$  we have*

$$\sup_{M>0} \lim_{C \rightarrow +\infty} \sup_{y \in \mathbb{R}^N} \int_{\{x \in \Omega_M; |x| \geq C\}} \frac{1}{(4\pi t)^{\frac{n}{2}}} \exp\left(-\frac{|x-y|^2}{4t}\right) dx < e^{\hat{\lambda}^1 t}. \quad (52)$$

Then  $\Delta_\Phi^{(1)}$  has a spectral gap.

*Proof:* Let  $\beta_{ess}^0$  be the essential lower bound of  $\Delta_\Phi^{(0)} + \lambda_\Phi(x)$ . By resumming the proofs of Theorem 35, one sees that (52) implies that the essential type of the semigroup generated by  $-(\Delta_\Phi^{(0)} + \lambda_\Phi(x))$  is strictly less  $\hat{\lambda}^1$  or equivalently  $\beta_{ess}^0 > -\hat{\lambda}^1 \geq -\lambda^1$ . Since  $\sigma(\Delta_\Phi^{(0)} + \lambda_\Phi(x)) \cap (-\infty, \beta_{ess}^0)$  is discrete, or equivalently  $\sigma((\Delta_\Phi^{(0)} + \lambda_\Phi(x)) \otimes Id) \cap (-\infty, \beta_{ess}^0)$  is discrete, then (51) and Theorem 58 show that  $\sigma(\Delta_\Phi^{(1)}) \cap (-\infty, \beta_{ess}^0)$  is discrete. The fact that  $\beta_{ess}^0 > -\lambda^1$  shows that  $\Delta_\Phi^{(1)}$  has a spectral gap. ■

## 9 On indefinite potentials

This last section continues the general theory of Section 2 and deals with *indefinite* potentials  $V = V_+ - V_-$ , i.e. we consider the more general case

$$“T - V” = “T - V_+ + V_-”$$

where  $T$  is the generator of a substochastic semigroup  $\{U(t); t \geq 0\}$  in  $L^1(\Omega; \mu)$  and  $V_\pm$  are nonnegative and unbounded and are *not* necessarily the positive and negative parts of  $V$ .

### 9.1 $L^1$ theory

We give first a meaning to “ $T - V_+ + V_-$ ”. Our general assumption is that  $V_+$  is admissible for  $\{U(t); t \geq 0\}$ , i.e.  $\{U_{V_+}(t); t \geq 0\}$  is a  $c_0$ -semigroup; its generator is then denoted by  $T_{V_+}$  (see the beginning of Section 2).

**Theorem 65** *Let  $V_- : D(T_{V_+}) \rightarrow L^1(\Omega; \mu)$  be  $T_{V_+}$ -bounded and let*

$$\lim_{\lambda \rightarrow +\infty} r_\sigma [V_-(\lambda - T_{V_+})^{-1}] < 1.$$

Then

- (i)  $T_{V_+} + V_- : D(T_{V_+}) \rightarrow L^1(\Omega; \mu)$  generates a positive semigroup  $\{W(t); t \geq 0\}$  on  $L^1(\Omega; \mu)$ .
- (ii)  $T_{V_+} + V_- : D(T_{V_+}) \rightarrow L^1(\Omega; \mu)$  is resolvent compact if (16) is satisfied (where  $\Omega_M$  are the sublevel sets of  $V_+$ ).

*Proof:* (i) The fact  $T_{V_+} + V_- : D(T_{V_+}) \rightarrow L^1(\Omega; \mu)$  generates a positive semigroup  $\{W(t); t \geq 0\}$  on  $L^1(\Omega; \mu)$  if and only if

$$\lim_{\lambda \rightarrow +\infty} r_\sigma [V_-(\lambda - T_{V_+})^{-1}] < 1$$

follows from a known result [22] (see e.g. [9] Chapter 5).

(ii) For  $\lambda$  large enough

$$(\lambda - T_{V_+} - V_-)^{-1} = (\lambda - T_{V_+})^{-1} \sum_{i=0}^{+\infty} (V_-(\lambda - T_{V_+})^{-1})^i$$

so that  $T_{V_+} + V_-$  is resolvent compact if  $T_{V_+}$  is. ■

**Corollary 66** *Let  $\lim_{\lambda \rightarrow +\infty} r_\sigma [V_-(\lambda - T_{V_+})^{-1}] < 1$  and let (16) be satisfied (where  $\Omega_M$  are the sublevel sets of  $V_+$ ).*

(i) *If  $T$  generates a holomorphic semigroup then  $T_{V_+} + V_-$  generates a (holomorphic) compact semigroup  $\{W(t); t \geq 0\}$  in  $L^1(\Omega; \mu)$ .*

(ii) *We assume that  $L^1(\Omega; \mu)$  is separable. Let  $\{U(t); t \geq 0\}$  be norm continuous and  $V_+$  be finite a.e. Let  $G_{V_+}(x, y)$  be the kernel of  $(1 - T_{V_+})^{-1}$  and let*

$$\sup_{y \in \Omega} \int_{\{V_\pm \geq j\}} G_{V_+}(x, y) V_\pm(x) \mu(dx) \rightarrow 0 \text{ as } j \rightarrow +\infty. \quad (53)$$

*Then  $T_{V_+} + V_-$  generates a compact semigroup  $\{W(t); t \geq 0\}$  in  $L^1(\Omega; \mu)$ .*

*Proof:* (i) We know that  $T_{V_+}$  generates a holomorphic semigroup [4][41]. The holomorphy of  $\{W(t); t \geq 0\}$  can be proved for instance as in ([59] Theorem 43). The compactness of  $\{W(t); t \geq 0\}$  is then deduced as in Theorem 11.

(ii) According to Remark 15 (ii),  $(1 - T)^{-1}$  has a kernel  $G(x, y)$ . Since (16) implies (15) then  $(1 - T_{V_+})^{-1}$  has also a kernel  $G_{V_+}(x, y)$ . By Theorem 12,  $T_{V_+}$  generates a norm continuous semigroup. A similar proof shows that  $T_{V_+} + V_-$  generates also a norm continuous semigroup and, similarly, the compactness of  $\{W(t); t \geq 0\}$  is deduced as in Theorem 11. ■

**Remark 67** (i) See Theorem 28 or Theorem 32 (i) to check (16) in metric measure spaces.

(ii) By the domination  $(1 - T_{V_+})^{-1} \leq (1 - T)^{-1}$  we may replace  $G_{V_+}(x, y)$  by  $G(x, y)$  in (53).

**Remark 68** The semigroup generated by  $T_{V_+} + V_-$  (under the general assumption  $\lim_{\lambda \rightarrow +\infty} r_\sigma [V_-(\lambda - T_{V_+})^{-1}] < 1$ ) is given by a Feynmann-Kac formula and is attached intrinsically to  $V$  (i.e. it is independent of the choice of a decomposition  $V = V_+ - V_-$ ), see [60] Remark 16.

The above compactness results are *due* to the part  $V_+$  of the potential. In particular, it is *not* possible that  $(\lambda - T_{V_+} - V_-)^{-1}$  be compact if  $(\lambda - T_{V_+})^{-1}$  is not ! This stems from the fact that the domination

$$(\lambda - T_{V_+} - V_-)^{-1} \geq (\lambda - T_{V_+})^{-1}$$

would imply the weak compactness of  $(\lambda - T_{V_+})^{-1}$  and in fact its compactness (see Lemma 6). Similarly  $\{e^{t(T_{V_+} + V_-^{nc})}; t \geq 0\}$  cannot be compact if  $\{e^{tT_{V_+}}; t \geq 0\}$  is not. On the other hand, regardless of  $V_+$ , the part  $V_-$  may induce different compactness results with different spectral consequences. To this end, we assume now that  $\Omega$  is a locally compact metric space endowed with a locally finite Borel measure  $\mu \geq 0$ . Let  $\Xi_n$  be a nondecreasing sequence of compact subsets such that  $\Omega = \cup_n \Xi_n$  and let

$$V_-^n := \begin{cases} V_- & \text{on } \Xi_n \\ 0 & \text{on } \Xi_n^c. \end{cases}$$

Let

$$V_-^{nc} := V_- - V_-^n = \begin{cases} V_- & \text{on } \Xi_n^c \\ 0 & \text{on } \Xi_n. \end{cases}$$

We note that  $V_- = V_-^n + V_-^{nc}$ . Our first assumption is

$$\lim_{n \rightarrow \infty} \sup_{y \in \Omega} \int_{\Xi_n^c} V_-(x) G_{V_+}(x, y) \mu(dx) < 1; \quad (54)$$

this expresses simply that  $\lim_{n \rightarrow +\infty} \|V_-^{nc}(\lambda - T_{V_+})^{-1}\|_{\mathcal{L}(L^1(\Omega))} < 1$ . Our second assumption is that for any compact sets  $C_1$  and  $C_2$  of  $\Omega$

$$\lim_{\mu(\Lambda) \rightarrow 0, \Lambda \subset C_1} \sup_{y \in C_2} \int_{\Lambda} G_{V_+}(x, y) V_-(x) \mu(dx) = 0. \quad (55)$$

We observe first that (54) implies that  $T_{V_+} + V_-^{nc} : D(T_{V_+}) \rightarrow L^1(\Omega)$  generates a positive semigroup. Secondly (55) expresses that  $V_-^n$  is  $T_{V_+}$ -weakly compact or, equivalently,  $V_-^n$  is  $(T_{V_+} + V_-^{nc})$ -weakly compact so that

$$\lim_{\lambda \rightarrow +\infty} r_\sigma [V_-^n(\lambda - T_{V_+} - V_-^{nc})^{-1}] = 0$$

and  $T_{V_+} + V_-^{nc} + V_-^n$  (i.e.  $T_{V_+} + V_-$ ) generates a positive semigroup (see [59] Theorem 4). Let  $s(T_{V_+} + V_-^{nc})$  be the spectral bound of  $T_{V_+} + V_-^{nc}$ . We define

$$\bar{s} := \lim_{n \rightarrow +\infty} s(T_{V_+} + V_-^{nc}).$$

Note that  $\{s(T_{V_+} + V_-^{nc})\}_n$  is nonincreasing and then

$$\bar{s} \geq s(T_{V_+}).$$

We give now an *upper bound* of the essential type of  $\{e^{t(T_{V_+} + V_-)}; t \geq 0\}$ .

**Theorem 69** *Let (54)(55) be satisfied. We assume that either  $\{e^{tT}; t \geq 0\}$  is holomorphic or  $\{e^{tT}; t \geq 0\}$  is norm continuous and (53) is satisfied. Then the essential type of  $\{e^{t(T_{V_+} + V_-)}; t \geq 0\}$  is less than or equal to  $\bar{s}$ .*

*Proof:* We know that if  $\{e^{tT}; t \geq 0\}$  is holomorphic then so are the semigroups  $\{e^{t(T_{V_+} + V_-^{nc})}; t \geq 0\}$  and  $\{e^{t(T_{V_+} + V_-)}; t \geq 0\}$  (see the arguments in the proof of Corollary 67). On the other hand, the inequality

$$\int_{\{V_-^{nc} \geq j\}} G_{V_+}(x, y) V_-^{nc}(x) \mu(dx) \leq \int_{\{V_- \geq j\}} G_{V_+}(x, y) V_-(x) \mu(dx)$$

shows that (53) is inherited by  $V_-^{nc}$  so that the two semigroups  $\{e^{t(T_{V_+} + V_-^{nc})}; t \geq 0\}$  and  $\{e^{t(T_{V_+} + V_-)}; t \geq 0\}$  are norm continuous. In both cases

$$]0, +\infty[ \ni t \rightarrow R(t) := e^{t(T_{V_+} + V_-)} - e^{t(T_{V_+} + V_-^{nc})} \text{ is norm continuous.} \quad (56)$$

We note that for  $\lambda$  large enough

$$\int_0^\infty [e^{t(T_{V_+} + V_-)} - e^{t(T_{V_+} + V_-^{nc})}] e^{-\lambda t} dt = (\lambda - T_{V_+} - V_-)^{-1} - (\lambda - T_{V_+} - V_-^{nc})^{-1}$$

and

$$\begin{aligned} (\lambda - T_{V_+} - V_-)^{-1} &= (\lambda - T_{V_+} - V_-^{nc} - V_-^n)^{-1} \\ &= (\lambda - T_{V_+} - V_-^{nc})^{-1} \sum_{i=0}^{+\infty} (V_-^n (\lambda - T_{V_+} - V_-^{nc})^{-1})^i \end{aligned}$$

so that

$$\int_0^\infty \left[ e^{t(T_{V_+} + V_-)} - e^{t(T_{V_+} + V_-^{nc})} \right] e^{-\lambda t} dt = (\lambda - T_{V_+} - V_-^{nc})^{-1} \sum_{i=1}^{+\infty} (V_-^n (\lambda - T_{V_+} - V_-^{nc})^{-1})^i$$

is weakly compact. By domination, for any  $\bar{t} > 0$  and  $\varepsilon > 0$ ,

$$\int_{\bar{t}}^{\bar{t}+\varepsilon} \left[ e^{t(T_{V_+} + V_-)} - e^{t(T_{V_+} + V_-^{nc})} \right] e^{-\lambda t} dt$$

is weakly compact and finally, thanks to (56),

$$\left[ e^{\bar{t}(T_{V_+} + V_-)} - e^{\bar{t}(T_{V_+} + V_-^{nc})} \right] e^{-\lambda \bar{t}} = \lim_{\varepsilon \rightarrow 0} \varepsilon^{-1} \int_{\bar{t}}^{\bar{t}+\varepsilon} \left[ e^{t(T_{V_+} + V_-)} - e^{t(T_{V_+} + V_-^{nc})} \right] e^{-\lambda t} dt$$

is also weakly compact since the limit holds in operator norm. It follows that  $\left\{ e^{t(T_{V_+} + V_-^{nc})}; t \geq 0 \right\}$  and  $\left\{ e^{t(T_{V_+} + V_-)}; t \geq 0 \right\}$  have the same essential spectrum and consequently the same essential type. Note that the essential type of  $\left\{ e^{t(T_{V_+} + V_-^{nc})}; t \geq 0 \right\}$  is less than or equal to its type and the latter coincides with the spectral bound  $s(T_{V_+} + V_-^{nc})$  since  $\left\{ e^{t(T_{V_+} + V_-^{nc})}; t \geq 0 \right\}$  is a positive semigroup in  $L^1(\Omega)$ . Finally the essential type of  $\left\{ e^{t(T_{V_+} + V_-)}; t \geq 0 \right\}$  is less than or equal to  $s(T_{V_+} + V_-^{nc})$  for *all*  $n$  and this ends the proof. ■

If we replace (54) by

$$\lim_{n \rightarrow \infty} \sup_{y \in \Omega} \int_{\Xi_n^c} V_-(x) G_{V_+}(x, y) \mu(dx) = 0 \quad (57)$$

then we obtain the following much more precise result:

**Theorem 70** *Let (55)(57) be satisfied. We assume that either  $\left\{ e^{tT}; t \geq 0 \right\}$  is holomorphic or  $\left\{ e^{tT}; t \geq 0 \right\}$  is norm continuous and (53) is satisfied. Then the essential type of  $\left\{ e^{t(T_{V_+} + V_-)}; t \geq 0 \right\}$  is equal to that of  $\left\{ e^{tT_{V_+}}; t \geq 0 \right\}$ . In particular if  $\left\{ e^{tT_{V_+}}; t \geq 0 \right\}$  has a spectral gap then  $\left\{ e^{t(T_{V_+} + V_-)}; t \geq 0 \right\}$  has also a spectral gap.*



*Proof:* We note that

$$V_-(\lambda - T_{V_+})^{-1} = V_-^n(\lambda - T_{V_+})^{-1} + V_-^{nc}(\lambda - T_{V_+})^{-1}$$

and (57) expresses that  $\lim_{n \rightarrow +\infty} \|V_-^{nc}(\lambda - T_{V_+})^{-1}\|_{\mathcal{L}(L^1(\Omega))} = 0$ . It follows that  $V_-(\lambda - T_{V_+})^{-1}$  is weakly compact since  $V_-^n(\lambda - T_{V_+})^{-1}$  is (as a consequence of (55)). Arguing as in the proof of Theorem 70, one sees that  $\int_0^\infty \left[ e^{t(T_{V_+} + V_-)} - e^{tT_{V_+}} \right] e^{-\lambda t} dt$  is weakly compact and deduce that the difference  $e^{t(T_{V_+} + V_-)} - e^{tT_{V_+}}$  is also weakly compact for  $t > 0$  which ends the proof of the first claim. Finally since the type of  $\left\{ e^{tT_{V_+}}; t \geq 0 \right\}$  is less than or equal to that of  $\left\{ e^{t(T_{V_+} + V_-)}; t \geq 0 \right\}$  then the latter has a spectral gap if the former has. ■

**Remark 71** *We have seen in Section 5 how to estimate the essential type of  $\left\{ e^{tT_{V_+}}; t \geq 0 \right\}$ .*

The stability of essential spectral bounds of perturbed generators holds under weaker assumptions:

**Theorem 72** *If (55)(57) are satisfied then  $s_{ess}(T_{V_+} + V_-) = s_{ess}(T_{V_+})$ . In particular if  $T_{V_+}$  has a spectral gap then  $T_{V_+} + V_-$  has also a spectral gap.*

*Proof:* The first claim follows from the weak compactness of the difference  $(\lambda - T_{V_+} - V_-)^{-1} - (\lambda - T_{V_+})^{-1}$  and the second claim from the standard inequality  $s(T_{V_+} + V_-) \geq s(T_{V_+})$ . ■

We complement Theorem 70 by:

**Theorem 73** *Let the assumptions in Theorem 70 be satisfied. If*

$$\sup_n \lim_{\lambda \rightarrow s(T_{V_+} + V_-^{nc})} r_\sigma \left[ V_-^n(\lambda - T_{V_+} - V_-^{nc})^{-1} \right] > 1$$

*then the essential type of  $\left\{ e^{t(T_{V_+} + V_-)}; t \geq 0 \right\}$  is less than its type, i.e. the semigroup  $\left\{ e^{t(T_{V_+} + V_-)}; t \geq 0 \right\}$  has a spectral gap.*

*Proof:* Suppose there exists  $n \in \mathbb{N}$  such that

$$\lim_{\lambda \rightarrow s(T_{V_+} + V_-^{nc})} r_\sigma \left[ V_-^n(\lambda - T_{V_+} - V_-^{nc})^{-1} \right] > 1.$$

We note that  $V_-^n(\lambda - T_{V_+} - V_-^{nc})^{-1}$  is positive so that

$$r_\sigma [V_-^n(\lambda - T_{V_+} - V_-^{nc})^{-1}] \in \sigma(V_-^n(\lambda - T_{V_+} - V_-^{nc}));$$

the weak compactness of  $V_-^n(\lambda - T_{V_+} - V_-^{nc})^{-1}$  implies that  $r_\sigma [V_-^n(\lambda - T_{V_+} - V_-^{nc})^{-1}]$  is an eigenvalue of  $V_-^n(\lambda - T_{V_+} - V_-^{nc})^{-1}$ . It follows that  $s(T_{V_+} + V_-^{nc})$ , the spectral bound of  $T_{V_+} + V_-^{nc}$ , is (strictly) less than that of

$$T_{V_+} + V_-^{nc} + V_-^n = T_{V_+} + V_-$$

and the latter is an eigenvalue of finite algebraic multiplicity (see [56] Chapter 5 for details). Finally, Theorem 70 implies that the essential type of  $\{e^{t(T_{V_+} + V_-)}; t \geq 0\}$  is (strictly) less than its type. ■

Similarly, we complement Theorem 71 by:

**Theorem 74** *Let the assumptions in Theorem 71 be satisfied. If*

$$\lim_{\lambda \rightarrow s(T_{V_+})} r_\sigma [V_-^n(\lambda - T_{V_+})^{-1}] > 1 \quad (58)$$

*then the essential type of  $\{e^{t(T_{V_+} + V_-)}; t \geq 0\}$  is less than its type, i.e. the semigroup  $\{e^{t(T_{V_+} + V_-)}; t \geq 0\}$  has a spectral gap.*

**Remark 75** *In the context of Theorem 73, (58) implies the existence of a spectral gap for perturbed generators, i.e.  $s_{ess}(T_{V_+} + V_-) < s(T_{V_+} + V_-)$ .*

## 9.2 $L^p$ theory

For the sake of simplicity (see Remark 82 below) we restrict ourselves to *symmetric* substochastic semigroups  $\{e^{tT}; t \geq 0\}$ , i.e. when  $\{e^{tT}; t \geq 0\}$  coincides with its dual  $\{e^{tT'}; t \geq 0\}$  (on  $L^\infty(\Omega)$ ) on the space  $L^1(\Omega) \cap L^\infty(\Omega)$ . Then  $\{e^{tT}; t \geq 0\}$  interpolates on all  $L^p(\Omega)$  ( $1 \leq p < \infty$ ) providing strongly continuous semigroups  $\{U_p(t); t \geq 0\}$  in  $L^p(\Omega)$  (where  $\{U_2(t); t \geq 0\}$  is self-adjoint in  $L^2(\Omega)$ ); we denote their generators by  $T_p$  (where  $T_2$  self-adjoint). We note that

$$U_{p,V_+}(t)f := \lim_{n \rightarrow +\infty} e^{t(T_p - V_{+n})}f$$

(where  $V_{+n} := \min\{V_+, n\}$ ) defines a positive semigroup which is strongly continuous in  $L^p(\Omega)$  if and only if  $\{U_{V_+}(t); t \geq 0\}$  is strongly continuous in

$L^1(\Omega)$  (see [83]); we denote this semigroup by  $\{U_{pV_+}(t); t \geq 0\}$  and its generator by  $T_{pV_+}$ . Moreover  $U_{2V_+}(t)$  is self-adjoint with (self-adjoint) generator  $T_{2V_+}$ . Under the general assumption

$$\lim_{\lambda \rightarrow +\infty} r_\sigma [V_-(\lambda - T_{V_+})^{-1}] < 1,$$

one shows (see [60]) that the semigroup  $\{e^{t(T_{V_+} + V_-)}; t \geq 0\}$  on  $L^1(\Omega)$ , with generator

$$T_{V_+} + V_- : D(T_{V_+}) \rightarrow L^1(\Omega),$$

interpolates on all  $L^p(\Omega)$  ( $1 \leq p < \infty$ ) providing positive strongly continuous semigroups  $\{W_p(t); t \geq 0\} = \{e^{tA_p}; t \geq 0\}$  in  $L^p(\Omega)$  (where  $A_2$  is self-adjoint in  $L^2(\Omega)$ ). We point out that  $V_-$  is *not* a priori  $T_{pV_+}$ -bounded for  $p > 1$ . However (see [63]),  $V_-$  is *form-bounded* with respect to  $-T_{2V_+}$  with relative form-bound less than or equal to

$$\lim_{\lambda \rightarrow +\infty} r_\sigma [V_-(\lambda - T_{V_+})^{-1}]$$

and

$$-A_2 = (-T_{2V_+}) \dot{+} (-V_-) \quad (\text{form-sum}).$$

**Theorem 76** *Let  $\lim_{\lambda \rightarrow +\infty} r_\sigma [V_-(\lambda - T_{V_+})^{-1}] < 1$  and (16) be satisfied (where  $\Omega_M$  are the sublevel sets of  $V_+$ ). Then  $\{W_p(t); t \geq 0\}$  is a compact semigroup in  $L^p(\Omega)$  for  $p > 1$ .*

*Proof:* We know (see Theorem 66 (ii)) that  $T_{V_+} + V_-$  is resolvent compact, i.e.  $(\lambda - T_{V_+} - V_-)^{-1}$  is compact in  $L^1(\Omega)$  for  $\lambda$  large enough. By interpolation  $(\lambda - A_p)^{-1}$  is compact in  $L^p(\Omega)$  for  $\lambda$  large enough. Since  $\{W_2(t); t \geq 0\} = \{e^{tA_2}; t \geq 0\}$  is selfadjoint then it is *norm continuous* so that, by interpolation,  $\{e^{tA_p}; t \geq 0\}$  are norm continuous too for  $p > 1$ . Then arguing e.g. as in the proof of Theorem 17, one sees that  $\{e^{tA_p}; t \geq 0\}$  are compact semigroups when  $p > 1$ . ■

**Remark 77** *Let  $\int_{B(x,1)} \frac{1}{1+V_+(y)} dy \rightarrow 0$  as  $|x| \rightarrow +\infty$  and let  $V_- \in L^p(\mathbb{R}^N)$  for some  $p > \frac{N}{2}$ . Then the fact that  $V_- \in L^p(\mathbb{R}^N)$  with  $p > \frac{N}{2}$  implies that  $\lim_{\lambda \rightarrow +\infty} \|V_-(\lambda - T_{V_+})^{-1}\|_{\mathcal{L}(L^1(\mathbb{R}^N))} = 0$  (see e.g. [77]) shows that  $\Delta - (V_+ - V_-)$  generates a compact semigroup in  $L^1(\mathbb{R}^N)$ . This result should be compared with that in [12] according to which  $-\Delta + V_+ - V_-$  has a discrete spectrum in  $L^2(\mathbb{R}^N)$  if  $\int_{B(x,1)} \frac{1}{1+V_+(y)} dy \rightarrow 0$  as  $|x| \rightarrow +\infty$  and  $V_- \in L^{\frac{N}{2}}(\mathbb{R}^N)$ .*

We deal now with spectral stability in  $L^p(\Omega)$ .

**Theorem 78** *Let (55)(57) be satisfied. We assume that either  $\{e^{tT}; t \geq 0\}$  is holomorphic or  $\{e^{tT}; t \geq 0\}$  is norm continuous and (53) is satisfied. Then  $\{e^{tA_p}; t \geq 0\}$  and  $\{e^{tT_{pV_+}}; t \geq 0\}$  have the same essential spectrum and consequently the same essential type. In particular,  $\{e^{tA_p}; t \geq 0\}$  has a spectral gap if  $\{e^{tT_{pV_+}}; t \geq 0\}$  has.*

*Proof:* We know that  $e^{t(T_{V_+} + V_-)} - e^{tT_{V_+}}$  is weakly compact in  $L^1(\Omega)$  for  $t > 0$  (see the proof of Theorem 71). In the new notations,  $e^{tA_1} - e^{tT_{1V_+}}$  is weakly compact in  $L^1(\Omega)$  and consequently is square  $\left[e^{tA_1} - e^{tT_{1V_+}}\right]^2$  is compact in  $L^1(\Omega)$ . It follows by interpolation that  $\left[e^{tA_2} - e^{tT_{2V_+}}\right]^2$  is compact in  $L^2(\Omega)$ ; the self-adjointness of  $e^{tA_2} - e^{tT_{2V_+}}$  implies the compactness of  $e^{tA_2} - e^{tT_{2V_+}}$  itself in  $L^2(\Omega)$  and then, by interpolation again,  $e^{tA_p} - e^{tT_{pV_+}}$  is compact in  $L^p(\Omega)$  for all  $p > 1$ . Thus  $\{e^{tA_p}; t \geq 0\}$  and  $\{e^{tT_{pV_+}}; t \geq 0\}$  share the same essential spectrum and then the same essential type. The fact that the spectral bound of  $T_{pV_+}$  is less than or equal to that of  $A_p$  ends the proof. ■

**Remark 79** (i) *We have dealt in Section 5 with spectral gaps of  $\{e^{tT_{pV_+}}; t \geq 0\}$ .*

(ii) *The existence of a spectral gap of  $\{e^{tA_p}; t \geq 0\}$  when  $\{e^{tT_{pV_+}}; t \geq 0\}$  has not a spectral gap is not dealt with here. We note also that it is unclear in general whether a spectral gap of  $\{e^{t(T_{V_+} + V_-)}; t \geq 0\}$  (see Theorem 75) can be inherited by  $\{e^{tA_p}; t \geq 0\}$ .*

(ii) *We can prove under (55)(57) alone that  $(\lambda - A_p)^{-1} - (\lambda - T_{pV_+})^{-1}$  is compact so that  $A_p$  and  $T_{pV_+}$  have the same essential spectrum and consequently the same essential spectral bound.*

**Remark 80** *We note that if  $\Omega = \mathbb{R}^n$  and if  $\{e^{tT}; t \geq 0\}$  admits a Gaussian estimate then  $\{e^{tT_{V_+}}; t \geq 0\}$  admits also a Gaussian estimate since  $e^{tT_{V_+}}$  is dominated pointwisely by  $e^{tT}$ . On the other hand, the assumption*

$$\lim_{\lambda \rightarrow +\infty} r_\sigma [V_-(\lambda - T_{V_+})^{-1}] < 1$$

*remains if we replace  $V_-$  by  $pV_-$  (the product of  $p$  and  $V_-$ ) with a suitable  $p > 1$  so that  $T_{V_+} + pV_-$  generates a positive semigroup and consequently*

(see [5] Theorem 3.6) the semigroup  $\{e^{t(T_{V_+}+V_-)}; t \geq 0\}$  admits a Gaussian estimate; it follows that the spectrum of  $\{e^{tA_p}; t \geq 0\}$  is  $p$ -independent, see e.g. [19].

**Remark 81** If  $\{e^{tT}; t \geq 0\}$  is not symmetric but its dual  $\{e^{tT'}; t \geq 0\}$  operates also on  $L^1(\Omega)$  with  $\lim_{\lambda \rightarrow +\infty} r_\sigma \left[ V_-(\lambda - T'_{V_+})^{-1} \right] < 1$  ( $L^1$  spectral radius) then one can show that  $\{e^{t(T_{V_+}+V_-)}; t \geq 0\}$  interpolates on all  $L^p(\Omega)$  ( $1 \leq p < \infty$ ) providing positive strongly continuous semigroups

$$\{W_p(t); t \geq 0\} = \{e^{tA_p}; t \geq 0\}$$

in  $L^p(\Omega)$  (in the spirit of [60] Theorem 19). If we resume the proof of Theorem 79, we have that  $R_p := e^{tA_p} - e^{tT_{pV_+}}$  (for  $p = 1$ ) is weakly compact in  $L^1(\Omega)$  so that (for  $|\mu|$  large enough),  $[(\mu - e^{tA_p})^{-1}R_p]^2$  is compact in  $L^p(\Omega)$  for all  $p \geq 1$ . Similarly,  $[(\mu - e^{tT_{pV_+}})^{-1}R_p]^2$  is compact in  $L^p(\Omega)$  for all  $p \geq 1$ . Then the analytic Fredholm alternative shows that  $e^{tA_p}$  and  $e^{tT_{pV_+}}$  have the same essential radius, see [82] Corollary 1.4. Finally they share the same essential type.

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